The Parameter Planes of $\lambda z^m \exp(z)$ for $m \geq 2$

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Abstract

We consider the families of entire transcendental maps given by $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ where $m \geq 2$. All functions $F_{\lambda,m}$ have a superattracting fixed point at $z = 0$, and a critical point at $z = -m$. In the parameter planes we focus on the capture zones, i.e., λ values for which the critical point belongs to the basin of attraction of $z = 0$, denoted by $A(0)$. In particular, we study the main capture zone (parameter values for which the critical point lies in the inmediate basin, $A^*(0)$) and prove that is bounded, connected and simply connected. All other capture zones are unbounded and simply connected. For each parameter λ in the main capture zone, $A(0)$ consists of a single connected component with non-locally connected boundary. For all remaining values of λ, $A^*(0)$ is a quasidisk. On a different approach, we introduce some families of holomorphic maps of $\mathbb{C}^*$ which serve as a model for $F_{\lambda,m}$, in the sense that they are related by means of quasiconformal surgery to $F_{\lambda,m}$.

1 Introduction and results

One of the central topics in complex dynamics is the study of the dynamics of the quadratic polynomial $Q_c(z) = z^2 + c$. The dynamical behavior of the map Q_c is determined by the orbit of the unique critical point $z = 0$. These maps have been thoroughly studied by many authors (see for example [DH1], [DH2], [CG], [M1], [L]). In analogy with the quadratic family of polynomials Q_c, the exponential map $E_\lambda(z) = \lambda \exp(z)$, with a unique asymptotic value at $v = 0$, is the simplest example of an entire transcendental map with rich and interesting dynamics.

The systematic study of cubic polynomials began with the work of Branner and Hubbard ([BH1]), who considered the two parameter family of monic and centered cubic polynomials which, after a suitable normalization, is given by $C_{a,b}(z) = z^3 - 3a^2 z + b$. Notice that any cubic polynomial is affine conjugate to one in this family. The dynamics of monic centered cubic polynomials is determined by the orbits of the two critical points located at $\pm a$. Moreover,

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they proved that the cubic connectedness locus, which is a subset of $\mathbb{C}^2$, consisting of all the parameters $(a, b) \in \mathbb{C} \times \mathbb{C}$ such that $\mathcal{J}(C_{a,b})$ is connected, is compact and connected. Many authors have investigated subfamilies, or slices, of the family of cubic polynomials (among others see [M], [Fau], [BH2], [R], [Z], [BuHe]).

Milnor studied the one parameter family of cubic polynomials having a superattracting fixed point ([M]). These polynomials are given by

$$M_a(z) = z^3 - \frac{3}{2} a z^2. \tag{1}$$

It is easy to see that M_a has a superattracting fixed point at $z = 0$, and a free critical point at $z = a$. When $z = a$ belongs to the basin of attraction of the superattracting fixed point $z = 0$ we say that the critical point $z = a$ has been captured. The connected components of the parameter space for which this phenomenon occurs are called *capture zones*. We also define the *main capture zone*, as the set of parameter values a for which the critical point $z = a$ belongs to the immediate basin of $z = 0$. The original parametrization of the Milnor cubic polynomials was $\tilde{C}_a(z) = z^3 - 3a^2 z + 2a^3 + a$, but both families are equivalent since they are conjugate under an affine change of coordinates.

Milnor ([M]) suggested two questions about the family of cubic polynomials M_a, one in the dynamical plane and another one in the parameter plane. The first one was to investigate wether for all parameter values a, the boundary of the immediate basin of attraction of $z = 0$ is a Jordan curve. The second one was to investigate wether the boundary of the main capture zone is a Jordan curve. Both questions were answered by Faught ([Fau]) using a modification of the Yoccoz's puzzle for a rational like mapping (see [R]). Faught proved that for all parameter values, $a \in \mathbb{C}$, the immediate basin of attraction of $z = 0$ is a Jordan domain and also the boundary of the main capture zone is a Jordan curve.

Roesch ([R]) generalized this result, in the dynamical plane, for an extension family of the Milnor cubic polynomial. More precisely, we can consider the family of polynomials

$$M_{m,a}(z) = z^{m+1} - \frac{m+1}{m} a z^m \tag{2}$$

as a generalized family of the Milnor cubic polynomials. For each $m \geq 2$ the point $z = 0$ is a superattracting fixed point of multiplicity m, and $z = a$ is a free critical point (when $m = 2$ we find exactly the Milnor cubic polynomial M_a). It is proven ([R]) that for every value of $m \geq 2$ and for all parameters $a \in \mathbb{C}$, the boundary of the immediate basin of attraction of the superattracting fixed point $z = 0$ is a Jordan curve.

Our goal in this work is to study some dynamical aspects of the families of entire transcendental maps

$$F_{\lambda,m}(z) = \lambda z^m \exp(z), \quad m \geq 2. \tag{3}$$

All functions of the form $F_{\lambda,m}$, with $m \geq 2$, have a superattracting fixed point at $z = 0$ of multiplicity m, which is also an asymptotic value. The only other critical point is $z = -m$. The coexistence of a superattracting fixed point and a free critical point makes this family an entire transcendental analogue of the generalized Milnor polynomials (Equation (2)).

Some functions in the family $F_{\lambda,m} = \lambda z^m \exp(z)$ for $m \geq 2$ have been used in the literature as examples of certain dynamical phenomena (see for example [Be], for a Baker domain at a positive distance from any singular orbit for a lift of a certain member $F_{\lambda,m}$). We also notice that fixed points of $F_{\lambda,m}$ appear in a different mathematical context. More precisely, $F_{\lambda,m}(z) = z$ is the characteristic equation of the following delay differential equation

$$\frac{d^{m-1}x}{dt^{m-1}} \;=\; \frac{1}{\lambda}\,x(t-1).$$

If we search for some value z_0 such that $x(t) = c\,e^{z_0 t}$ is a solution, we obtain the characteristic equation $\lambda z_0^m \exp(z_0) \;=\; z_0$.

In ([FG]) we made an initial study of the discrete dynamical system generated by the map $F_{\lambda,m}$. We focussed our attention in a description of the dynamical planes, and specially on the basin of attraction of the superattracting fixed point at $z = 0$. In this paper we turn our attention to the parameter planes of the family of functions $F_{\lambda,m}$. As usual in complex dynamics as A. Douady dixit: "you first plow in the dynamical plane and then harvest in the parameter space".

As we mentioned, the origin is a superattracting fixed point of the function $F_{\lambda,m}$, for all $m \geq 2$ and $\lambda \in \mathbb{C}$. We denote by $A(0) = A_{\lambda,m}(0)$ the basin of attraction of the origin, given by

$$A(0) = A_{\lambda,m}(0) = \{z \in \mathbb{C},\, F^{\circ n}_{\lambda,m}(z) \to 0 \text{ as } n \to \infty\}. \tag{4}$$

The immediate basin of attraction of $z = 0$ is the connected component of $A(0)$ containing $z = 0$, and we denote it by $A^*(0) \;=\; A^*_{\lambda,m}(0)$.

One of the main objective of this work is the study of $A_{\lambda,m}(0)$. We would like to answer the following questions: How many connected components does $A_{\lambda,m}(0)$ have? Are they simply connected? Are they bounded? When is the boundary of $A^*_{\lambda,m}(0)$ locally connected?

For some parameter values, the free critical point $z = -m$ belongs to the basin of attraction of $z = 0$, in which case we say that it has been captured. The connected components of parameter space for which this phenomenon occurs are called *capture zones*, and they clearly do not exist for members of the family $F_{\lambda,m}$ with $m < 2$, i.e., for the exponential family.

We will study the capture zones given by

$$\mathcal{H}^n_m = \{\lambda \in \mathbb{C} \,|\, F^n_{\lambda,m}(-m) \in A^*_{\lambda,m}(0) \text{ and } n \text{ is the smallest number with this property}\} \tag{5}$$

As a special case, we define the *main capture zone*, $\mathcal{H}^0_m$, as the set of parameter values λ for which the critical point $z = -m$ itself belongs to the immediate basin of 0. That is,

$$\mathcal{H}^0_m \;=\; \{\lambda \in \mathbb{C} \,|\; -m \in A^*_{\lambda,m}(0)\}. \tag{6}$$

We shall see that, this is a quite special capture zone since its boundary separates the parameter values for which $\mathcal{F}(F_{\lambda,m})$ has one connected component from those for which it has infinitely many.

In the parameter plane we will answer the following questions: Is $\mathcal{H}^n_m$ connected? Are the connected components of $\mathcal{H}^n_m$ simply connected? Are they bounded? How does the boundary of $A^*_{\lambda,m}(0)$ depend on λ? Is $\partial A^*_{\lambda,m}(0)$ locally connected when λ belongs to $\mathcal{H}^n_m$?

In order to answer all of these questions we divide our study into two parts. In the first one, we study directly the family of functions $F_{\lambda,m} = \lambda z^m \exp(z)$ using standard tools in complex dynamics. In the second one, we relate it to a new family of maps given by

$$G_{\alpha,\beta,m}(z) = \exp(i\alpha) z^m \exp(\beta/2(z - 1/z))\,, \tag{7}$$

where α and β are real numbers and $m \geq 2$. The family of functions $G_{\alpha,\beta,2}$ have been investigated as real maps on the unit circle by M. Misiurewicz and A. Rodrigues ([MR]). Using quasiconformal surgery, we relate members of $G_{\alpha,\beta,m}$ to those of $F_{\lambda,m}$, and use this correspondence to prove some results for the original maps.

We first concentrate on the dynamical plane and especially in the basin of attraction $A_{\lambda,m}(0)$. More precisely, we prove the following result related to the topology of the connected components of $A_{\lambda,m}(0)$.

Proposition A. *Let $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$ and $F_{\lambda,m}(z) = \lambda z^m \exp(z)$. Let $A_{\lambda,m}(0)$ and $A^*_{\lambda,m}(0)$ be the basin and the immediate basin of attraction of $z = 0$ for the map $F_{\lambda,m}$, respectively. The following statements hold.*

a) All connected components of the Fatou set of $F_{\lambda,m}$ are simply connected.

b) $A_{\lambda,m}(0)$ has either one or infinitely many connected components.

c) All the connected components of $A_{\lambda,m}(0)$ different from $A^*_{\lambda,m}(0)$ are unbounded.

Further we describe the main features of the parameter planes of the functions $F_{\lambda,m}$ and, in particular, the structure of the capture zones. We summarize some of these facts in the following theorems. In the first one we study the topology of the capture zones. In the second one we investigate the local connectivity of the boundary of $A^*_{\lambda,m}(0)$. In the third we study the complement of the closure of the main capture zone $\mathcal{H}^0_m$.

Theorem B. *For all parameters $m \in \mathbb{N}$, $m \geq 2$, let $\mathcal{H}^n_m$, $\mathcal{H}^0_m$ be the capture zones as in (5) and (6), respectively. The following statements hold.*

a) The critical point $-m$ belongs to $A^*_{\lambda,m}(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*_{\lambda,m}(0)$. Hence $\mathcal{H}^1_m = \emptyset$.

b) There exist $\rho = \rho(m)$, $\rho' = \rho'(m)$ verifying $0 < \rho < \rho'$ such that $D_\rho \subset \mathcal{H}^0_m \subset D_{\rho'}$, where $D_r = \{z \in \mathbb{C} \,|\, |z| < r\}$.

c) The main capture zone $\mathcal{H}^0_m$ is connected and simply connected.

d) Let $n \geq 2$. All the connected components of $\mathcal{H}^n_m$ are simply connected and unbounded.

Theorem C. *Let $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$. Let $\mathcal{H}^n_m$, $\mathcal{H}^0_m$ be the capture zones as in (5) and (6), respectively. The following statements hold.*

a) If $\lambda \in \mathcal{H}^0_m$ then $A_{\lambda,m}(0) = A^*_{\lambda,m}(0)$. However if $\lambda \notin \mathcal{H}^0_m$ then $A_{\lambda,m}(0)$ has infinitely many connected components.

b) If $\lambda \in \mathcal{H}_m^0$ the boundary of $A^*_{\lambda,m}(0)$ (which is equal to the Julia set) is a Cantor bouquet not locally connected.

c) Let $\mathcal{U}_m$ be the unbounded connected component of $\mathbb{C}\setminus\overline{\mathcal{H}_m^0}$. If $\lambda \in \mathcal{U}_m$, then the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle. In particular, if $\lambda \in \mathcal{H}_m^n$ for any $n \geq 2$ the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle.

Theorem D. *Let $\mathcal{U}_m$ be the unbounded connected component of $\mathbb{C}\setminus\overline{\mathcal{H}_m^0}$. The following statements hold.*

a) $\partial\mathcal{H}_m^0 = \partial\mathcal{U}_m$.

b) If there exist a bounded connected component $\mathcal{V}$ of $\mathbb{C}\setminus\overline{\mathcal{H}_m^0}$, then $\mathcal{U}_m, \mathcal{H}_m^0$ and $\mathcal{V}$ are lakes of Wada, i.e., they have a common boundary.

There is another question which will remain unanswered in this work and which we state as a conjecture.

Conjecture. The boundary of $\mathcal{H}_m^0$ is a Jordan curve.

Finally, we take a second approach, using quasiconformal surgery, to further describe the maps at hand. More precisely,

$$G_{\alpha,\beta,m}(z) = \exp(i\alpha)z^m \exp(\beta/2(z-1/z)),$$

where α and β are real numbers and $m \geq 2$, which we relate to the original one by means of quasiconformal surgery. Roughly speaking quasiconformal surgery is a technique to construct holomorphic maps with some prescribed dynamics. In our case, we combine two dynamical systems acting in different parts of the plane to construct a new system that combines the dynamics of both. In this process we use quasiconformal mappings to glue different behaviors. The key ingredient of this technique is to use the Measurable Riemann Mapping Theorem ([Ah, LV]) in order to assure that the corresponding mapping is a holomorphic map.

For our construction, it will play a fundamental role the fact that $G_{\alpha,\beta}$ preserves the unit circle S^1. More precisely, $G_{\alpha,\beta}$ induces a one dimensional mapping on the unit circle

$$\widetilde{G}_{\alpha,\beta,m} : \theta \to \alpha + m\theta + \beta\sin(\theta) \qquad \text{mod } (2\pi)\,, \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

We are interested in the set of parameters

$$\mathcal{W}_m = \{\alpha, \beta \,|\, \widetilde{G}_{\alpha,\beta,m} \text{ is quasisymmetrically conjugate to } \theta \mapsto m\theta\}$$

which in particular includes all those for which $\widetilde{G}_{\alpha,\beta,m}$ is an expanding map on the unit circle ([SS]). This is summarized in the following theorem.

Theorem E. *For any $(\alpha,\beta) \in \mathcal{W}_m$, there exist a λ in the complement of $\overline{\mathcal{H}_m^0}$ such that $F_{\lambda,m}$ is quasiconformally conjugate on the complement of $A^*_{\lambda,m}(0)$ to $G_{\alpha,\beta}$ on the complement of the closed unit disc. For this value of λ the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle.*

The rest of the paper is organized as follows. In Section 2 we present some previous results concerning the basin of attraction of the origin. In Section 3 we summarize some tools which we will use in this paper. Finally, Sections 4 and 5 are devoted to prove the main results of this work. Experts can read directly Section 4.

2 Preliminaries

The systematic study of the functions $F_{\lambda,m}$ was started in [FG]. In this section we recall some results from that work that will be useful later on.

Theorem 2.1 (Skeleton of $A_{\lambda,m}(0)$, see Figure 1). *Let $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, $m \geq 2$ and $F_{\lambda,m}(z) = \lambda z^m \exp(z)$. Let $A^*_{\lambda,m}(0)$ be the immediate basin of attraction of $z = 0$ for the map $F_{\lambda,m}$. The following statements hold.*

a) For $\lambda \neq 0$, if we define $\epsilon_0 = \epsilon_0(|\lambda|, m) > 0$ as the unique positive solution of $x^{m-1}e^x = 1/|\lambda|$; then $A^*_{\lambda,m}(0)$ contains the disk $D_{\epsilon_0} = \{z \in \mathbb{C}\, ;\, |z| < \epsilon_0\}$.

b) There exist $x_0 = x_0(|\lambda|, m) < 0$ and a continuous (decreasing) function $x \mapsto C(x) > 0$ defined for $x < x_0$ such that the open set

$$H_{|\lambda|,m} = \left\{ z = x + yi \;\middle|\; \begin{array}{ll} x & \in (-\infty, x_0) \\ y & \in (-C(x), C(x)) \end{array} \right\}$$

satisfies $F_{\lambda,m}\left(H_{|\lambda|,m}\right) \subset D_{\epsilon_0}$.

c) There exist infinitely many strips, denoted by $S^k_{\lambda,m}$, which are preimages of $H_{|\lambda|,m}$. These horizontal strips extend to $+\infty$, and they have asymptotic width equal to π.

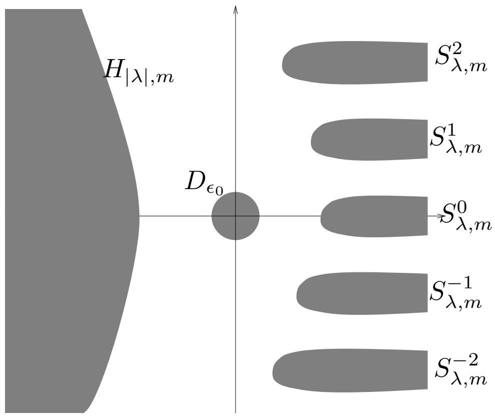


Figure 1: Sketch of some sets included in the basin of attraction of $z = 0$. Precisely, $D_{\epsilon_0} \subset A^*_{\lambda,m}(0)$, $F_{\lambda,m}(H_{|\lambda|,m}) \subset D_{\epsilon_0}$ and $F_{\lambda,m}\left(S^k_{\lambda,m}\right) \subset H_{|\lambda|,m}$.

The skeleton of the main components of $A_{\lambda,m}(0)$ is needed to study later the parameter planes. In the first statement of Theorem 2.1 we give an estimate of the size of the immediate basin of attraction of $z = 0$. Since $z = 0$ is a superattracting fixed point, there exists $\epsilon_0 > 0$ such that the open disk $D_{\epsilon_0} = \{z \in \mathbb{C}\, ;\, |z| < \epsilon_0\}$ is contained in the immediate basin of attraction of $z = 0$. In the second statement we find the first preimage D_{ϵ_0}, which contains an unbounded open set in $\mathbb{C}$ extending to the left and containing an unbounded interval $(-\infty, x_0)$ for some real value x_0. In the third statement we find the second preimage of D_{ϵ_0}, which contains countably many horizontal strips extending to $+\infty$.

In the following auxiliary result we find a lower bound for ϵ_0, which will be used later on.

Lemma 2.2. *The value of* ϵ_0 *is always larger than or equal to* $\min\{1, (\frac{1}{|\lambda|e})^{\frac{1}{m-1}}\}$

3 Tools

In this section we present well known tools in complex dynamics which we will use in this paper. We also present applications of some of them to our particular case. The first tool is a classical result related to the behavior of holomorphic maps near a superattracting fixed point ([Bo]), which we apply to make a detailed description of the superattracting basin of $z = 0$ for $F_{\lambda,m}$. The second section is related to the extension of a Holomorphic motion, established by Słodkowski ([Sl]). In the third one, we recall shortly the relevant definitions and results relative to quasiconformal mappings ([Ah], [LV]). Finally, in the miscelanea section we provide precise definitions of several concepts related to circle maps.

3.1 Böttcher coordinates near a superattracting fixed point

Theorem 3.1. *Suppose that* f *is an holomorphic map, defined in some neighborhood* U *of* 0*, having a superattracting fixed point at* 0*, i.e.,*

$$f(z) = a_m\, z^m + a_{m+1}\, z^{m+1} + \cdots \text{ where } m \geq 2, \text{ and } a_m \neq 0.$$

Then, there exists a local conformal change of coordinate $w = \varphi(z)$*, called Böttcher coordinate at* 0 *(or Böttcher map), such that* $\varphi \circ f \circ \varphi^{-1}$ *is the map* $w \to w^m$ *throughout some neighborhood of* $\varphi(0) = 0$*. Furthermore,* φ *is unique up to multiplication by an* $(m-1)$*-st root of unity.*

In practice, it is customary to make a linear change of coordinates so that the map f is monic, i.e., so that $a_m = 1$. When f is monic we obtain a unique Böttcher coordinate such that $\lim_{z\to 0} \frac{\varphi(z)}{z} = 1$. Also it is natural to extend φ to a maximal domain using the functional relation $\varphi(f(z)) = \varphi(z)^m$ (see, [DH1, DH2] or [BuHe] for details). One might hope that the change of coordinates $z \mapsto \varphi(z)$ extends throughout the entire immediate basin of attraction of the superattractive point as a holomorphic mapping. However, this is not always possible. Such an extension involves computing expressions of the form

$$z \mapsto \sqrt[m]{\varphi(f(z))},$$

and this does not work in general since the $n-$th root cannot be defined as a single valued function. For example, when some other point in the basin maps exactly onto the superattracting point, or when the basin is not simply connected.

Using the Böttcher map we can define a useful polar coordinate near 0. We define the *dynamical ray of argument* θ, where $\theta \in \mathbb{R}/\mathbb{Z}$, to be the image under the inverse of the Böttcher map of the half line through 0 with argument θ turns, *i.e.* $2\pi\theta$ radians,

$$R_0(\theta) = \varphi^{-1}(\{se^{2\pi i\theta} \mid s \geq 0\}).$$

We say that the *dynamical ray* $R_0(\theta)$ *lands* if and only if there exist

$$\lim_{s\to 1} \varphi^{-1}(se^{2\pi i\theta}).$$

When a dynamical ray $R_0(\theta)$ lands we call the limit the *landing point* of the ray $R_0(\theta)$.

We define the *dynamical equipotential of level* s, where $0 < s < 1$, to be the image under the inverse of the Böttcher map of the circle of radius s and centered at 0,

$$E_0(s) = \varphi^{-1}(\{se^{2\pi i\theta} \,|\, 0 \leq \theta < 1\}).$$

Since φ conjugates f to $w \to w^m$, the dynamics under f is easy to compute on these dynamical objects (rays and equipotentials). Precisely, we have

$$f(R_0(\theta)) \;=\; R_0(m\theta) \qquad \text{and} \qquad f(E_0(s)) \;=\; E_0(s^m)$$

As we already mentioned, the Böttcher map verifies the functional equation or Böttcher equation

$$\varphi(f(z)) = \varphi(z)^m$$

On the other hand, there exists an explicit form of the Böttcher map, given by

$$\varphi(z) = \lim_{n\to\infty} (f^{\circ n}(z))^{1/m^n}$$

In order to remove the ambiguity of the root, we write the sequence in the following form:

$$\varphi(z) = z \cdot \left[\frac{f(z)}{z^m}\right]^{1/m} \cdot \left[\frac{f^{\circ 2}(z)}{(f(z))^m}\right]^{1/m^2} \cdots \left[\frac{f^{\circ n}(z)}{(f^{\circ(n-1)}(z))^m}\right]^{1/m^n} \cdots \tag{8}$$

For the general term, we have

$$\left[\frac{f^{\circ n}(z)}{(f^{\circ(n-1)}(z))^m}\right]^{1/m^n} = \left[1 + \mathcal{O}(f^{\circ(n-1)}(z))\right]^{1/m^n}$$

Hence, in a neighborhood of the superattracting fixed point $z = 0$, we can define the root by the binomial formula:

$$(1+u)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} u^n \text{ when } |u| < 1.$$

It is not difficult to see that the product converges uniformly.

In our case, $z = 0$ is a superattracting fixed point of $F_{\lambda,m} = \lambda z^m \exp(z)$. Using a suitable linear change of variables we obtain a new family of entire transcendental maps, so that near the superattracting fixed point $z = 0$, the functions can be written as $z^m + O(z^{m+1})$, and thus have a preferred Böttcher coordinate in this region. More precisely, we consider the following auxiliary family of entire transcendental maps,

$$L_{a,m}(z) \;=\; z^m e^{z/a}, a \in \mathbb{C} \setminus \{0\}, \text{ and } m \in \mathbb{N}, m \geq 2. \tag{9}$$

In the next lemma we prove some fundamental properties of the Böttcher coordinate near $z = 0$ for the map $L_{a,m}$. In particular, we obtain an explicit expression of the Böttcher map and we see that it extends to the whole immediate basin of attraction of $z = 0$.

Lemma 3.2. *Consider $L_{a,m}(z) = z^m \exp(z/a)$ for $a \neq 0$ and $m \geq 2$. Then, the Böttcher coordinate φ_a extends to the whole immediate basin of attraction of the superattracting fixed point $z = 0$.*

Proof. The map $L_{a,m}$ is affine conjugate to $F_{\lambda,m}$ with $\lambda = a^{m-1}$ through the map $c_a(z) = az$. In other words, if we choose two parameter values λ_0 and a_0 such that $\lambda_0 = a_0^{m-1}$, then $F_{\lambda_0,m}$ and $L_{a_0,m}$ are conformally conjugate, i.e.

$$L_{a_0,m}(z) = (c_{a_0}^{-1} \circ F_{\lambda_0,m} \circ c_{a_0})(z) \qquad \forall z \in \mathbb{C}.$$

For each $a \neq 0$, and when z is small enough we can write the Böttcher coordinate $\varphi_a(z)$ using the auxiliary expression (8). More precisely, we have

$$\varphi_{a,m}(z) = z \cdot \left[\frac{L_{a,m}(z)}{z^m}\right]^{1/m} \cdot \left[\frac{L_{a,m}^{\circ 2}(z)}{(L_{a,m}(z))^m}\right]^{1/m^2} \cdots \left[\frac{L_{a,m}^{\circ n}(z)}{(L_{a,m}^{\circ(n-1)}(z))^m}\right]^{1/m^n} \cdots$$

For the general term, we have

$$\left[\frac{L_{a,m}^{\circ n}(z)}{(L_{a,m}^{\circ(n-1)}(z))^m}\right]^{1/m^n} = \exp\left[\frac{L_{a,m}^{\circ(n-1)}(z)}{a\, m^n}\right].$$

Hence, in a neighborhood of the superattracting fixed point $z = 0$, we obtain

$$\varphi_a(z) \; = \; z \exp\left[\sum_{n=0}^{\infty} \frac{L^{\circ n}(z)}{a\, m^{n+1}}\right]. \tag{10}$$

Finally, we observe that this holomorphic map is well defined (the series converges) in the whole immediate basin of attraction of $z = 0$.

□

3.2 Holomorphic motions

Definition. Let $X \subset \hat{\mathbb{C}}$ we say that a map

$$\begin{array}{rccl} \Phi: & \mathbb{D} \times X & \rightarrow & \mathbb{D} \times \hat{\mathbb{C}} \\ & (c, z) & \rightarrow & \Phi(c,z) = (c, \Phi_c(z)) = (c, \Phi^z(c)) \end{array}$$

is a *holomorphic motion* of X parameterized by $\mathbb{D}$ if

(a) $\Phi_0(z) = z$ for all $z \in X$.

(b) $\Phi_c(z)$ is injective for all fixed $c \in \mathbb{D}$.

(c) For all $z \in X$, the map $\Phi^z : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic.

There are two important theorems studying the extension of a holomorphic motion. The first one is the λ *Lemma* ([MSS]) and it extends a holomorphic motion of X to the closure of X. The second one is *Słodkowski* Lemma ([Sl]) and it extends a holomorphic motion parameterized in $\mathbb{D}$ to the whole Riemann sphere. We only recall the Słodkowski Lemma, since it is a generalization of the λ Lemma.

Theorem 3.3 (Słodkowski Lemma, [Sl]). *Let $\Phi : \mathbb{D} \times X \to \mathbb{D} \times \hat{\mathbb{C}}$ be a holomorphic motion. Then, we can extend Φ to a holomorphic motion $\widetilde{\Phi} : \mathbb{D} \times \hat{\mathbb{C}} \to \mathbb{D} \times \hat{\mathbb{C}}$. Moreover, for every parameter $c \in \mathbb{D}$, the map $\widetilde{\Phi}_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal homeomorphism whose dilatation ratio K_c is bounded by $\frac{1+|c|}{1-|c|}$.*

In the following lemma we prove that the holomorphic motion of a quasidisk is also a quasidisk. This property will play a fundamental role to prove Theorem C.

Lemma 3.4. *Let $\mathcal{U}$ be a quasidisk, i.e., assume that there exist a quasiconformal mapping $h : \mathbb{C} \to \mathbb{C}$ so that $\mathcal{U} = h(\mathbb{D})$. Let $\Phi : \mathbb{D} \times \mathcal{U} \to \mathbb{D} \times \mathbb{C}$ be a holomorphic motion of $\mathcal{U}$. Then for all $c \in \mathbb{D}$ we have that $\Phi_c(\mathcal{U})$ is also a quasidisk.*

Proof. Applying the Słodkowski Lemma (Theorem 3.3) we can extend Φ to a holomorphic motion $\widetilde{\Phi} : \mathbb{D} \times \hat{\mathbb{C}} \to \mathbb{D} \times \hat{\mathbb{C}}$ such that for every parameter $c \in \mathbb{D}$, the map $\widetilde{\Phi}_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal mapping. If we denote by $\mathcal{U}_c := \{\Phi(c, z) \mid z \in \mathcal{U}\}$, we have that $\widetilde{\Phi}_c \circ h : \mathbb{C} \to \mathbb{C}$ is a quasiconformal mapping and $\overline{\mathcal{U}_c} = \widetilde{\Phi}_c \circ h(\overline{\mathbb{D}})$. □

3.3 Quasiconformal surgery

Definition. A *quasiconformal map* of $\mathbb{C}$ is a homeomorphism φ such that small infinitesimal circles are mapped onto small infinitesimal ellipses of bounded axes ratio. The analytic formulation of this condition is that $\varphi(x + iy)$ is absolutely continuous in x for almost every y and in y for almost every x and that the partial derivatives are locally square integrable and satisfy the *Beltrami differential equation*

$$\frac{\partial \varphi}{\partial \overline{z}} = \mu(z) \frac{\partial \varphi}{\partial z} \text{ for almost all } z \in \mathbb{C},$$

where μ is a complex measurable function with

$$|\mu(z)| \leq \kappa < 1 \text{ for } z \in \mathbb{C}.$$

In this case we say that φ is $\kappa - quasiconformal$.

An *almost complex structure* σ on $\mathbb{C}$ is a measurable field of ellipses $(E_z)_{z \in \mathbb{C}}$, equivalently defined by a measurable Beltrami form μ on $\mathbb{C}$

$$\mu = u \frac{d\bar{z}}{dz}.$$

The correspondence between Beltrami forms and complex structures is as follows: the argument of $u(z)$ is twice the argument of the major axis of E_z, and $|u(z)| = \frac{K-1}{K+1}$ where $K \geq 1$ is the ratio of the lengths of the axes.

The *standard complex structure* σ_0 is defined by circles or by the Beltrami form $\mu_0 = 0$.

Suppose that $\varphi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal homeomorphism. Then φ gives rise to an *almost complex structure* σ on $\mathbb{C}$. For almost every $z \in \mathbb{C}$, φ is differentiable and the $\mathbb{R}-$linear tangent map $T_\varphi : T_z\mathbb{C} \to T_{\varphi(z)}\mathbb{C}$ defines, up to multiplication by a positive factor, an ellipse E_z in $T_z\mathbb{C}$:

$$E_z = (T_z\varphi)^{-1}(S^1).$$

Moreover, there exists a constant $K > 1$ such that the ratio of the axes of E_z is bounded by K for almost every $z \in \mathbb{C}$. The smallest bound is called the *dilatation ratio* of φ.

Equivalently, φ defines a measurable Beltrami form on $\mathbb{C}$

$$\mu = \frac{\overline{\partial}\varphi}{\partial\varphi} = \frac{\frac{\partial\varphi}{\partial\bar{z}}}{\frac{\partial\varphi}{\partial z}}\frac{d\bar{z}}{dz} = u(z)\frac{d\bar{z}}{dz}.$$

An almost complex structure is quasiconformally equivalent to the standard structure if it is defined by a measurable field of ellipses with bounded dilatation ratio.

Given $\varphi : \mathbb{C} \to \mathbb{C}$ a quasiconformal homeomorphism, an almost complex structure σ on $\mathbb{C}$ can be *pulled back* into an almost complex structure $\varphi^*\sigma$ on $\mathbb{C}$. If σ is defined by an infinitesimal field of ellipses $(E_z)_{z\in\mathbb{C}}$, then $\varphi^*\sigma$ is defined by $(E_z)_{z\in\mathbb{C}}$ where $E_z = (T_z\varphi)^{-1}\, E_{\varphi(z)}$ whenever defined.

To *integrate* an almost complex structure σ means to find a quasiconformal homeomorphism φ such that $(T_z\varphi)^{-1}(S^1) = \rho(z)E_z$ for almost every $z \in \mathbb{C}$. Informally, we will say that σ is transported to σ_0 by σ.

Surgery techniques are based on the following result:

Theorem 3.5 (Measurable Riemann mapping Theorem, [Ah, LV]). *Let σ_μ be any almost complex structure on $\mathbb{C}$ given by the Beltrami form*

$$\mu = u\frac{d\bar{z}}{dz}$$

with bounded dilatation ratio, i.e.,

$$||\mu||_\infty := \sup |u(z)| < m < 1.$$

Then σ_μ is integrable, i.e., there exists a quasiconformal homeomorphism φ such that

$$\mu = \frac{\overline{\partial}\varphi}{\partial\varphi},$$

or equivalently $\varphi^*\sigma_0 = \sigma_\mu$. Moreover, $\varphi : \mathbb{C} \to \mathbb{C}$ is unique up to composition with an affine map.

Remark 3.6. The application of Ahlfors-Bers' theorem to complex dynamics is the following. Let f and σ_μ be, a quasiregular mapping of $\mathbb{C}$ and an almost complex structure with bounded dilatation ratio, such that $f^*\sigma_\mu = \sigma_\mu$. If we apply Theorem 3.5 to integrate σ_μ, we obtain a quasiconformal mapping φ such that $\varphi^*\sigma_0 = \sigma_\mu$. Then $g = \varphi \circ f \circ \varphi^{-1}$ verifies $g^*\sigma_0 = \sigma_0$, and hence g is a holomorphic map of $\mathbb{C}$. Moreover, f and g are quasiconformally conjugate, i.e., they have the same dynamics.

3.4 Miscellanea

Our goal in this subsection is to make precise definitions of expanding maps ([dMvS]), and the quasiconformal extension of a quasisymmetric map on the circle ([Pom]). We also need the concept of growth order of a continuous function.

Definition. We say that a C^1 map $f : \mathbb{T} \to \mathbb{T}$ is *expanding* if there exist real constants $C > 0$ and $\mu > 1$ such that

$$|D(f^{\circ n}(x))| \; > \; C\mu^n$$

for all $n \in \mathbb{N}$ and all $x \in \mathbb{T}$.

We observe that a sufficient condition to assure that f is expanding is given by

$$\min\{|f'(x)|\,,\, x \in \mathbb{T}\} > 1.$$

The following theorem states that any two expanding maps of the same degree are quasisymmetrically conjugate.

Theorem 3.7 (Shub and Sullivan, [SS]). *Let $f, g : \mathbb{T} \to \mathbb{T}$ be expanding and $C^{1+\delta}$, with $\delta \in (0, 1)$, maps of degree m. Then there exist a quasisymmetric conjugacy $\varphi : \mathbb{T} \to \mathbb{T}$ such that $f = h^{-1} \circ g \circ h$.*

Quasisymmetry is precisely the property that allows a circle maps to be extendable to a quasiconformal map of the disc, as shown by the following theorem.

Theorem 3.8 (Beurling and Ahlfors [BA], Douady and Earle [DE]). *Let $h : \mathbb{T} \to \mathbb{T}$ be an orientation preserving quasisymmetric map. We can extend h to a quasiconformal map $\widehat{H} : \mathbb{D} \to \mathbb{D}$. Moreover, if $\sigma, \tau \in M\ddot{o}b(\mathbb{D})$ then the extension of $\sigma \circ h \circ \tau$ is given by $\sigma \circ \widehat{H} \circ \tau$*

Finally we will need the definition of the growth order of a continuous function.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. We define $M(r, f) := \max_{|z|=r} |f(z)|$ and the *growth order* $\rho(f)$ by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, f)}{\log r}$$

where $\log^+(t) \; = \; \log(\max(1, t))$

4 Transcendental part

When we consider a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ with an essential singularity at infinity, this point plays a crucial role. For instance, the little Picard Theorem says that an entire function assumes every value in the complex plane with at most one exception, in any neighborhood of infinity. Thus, in general, iteration of entire transcendental maps is more complicated than rational maps. As an example, there are transcendental maps presenting wandering domains ([B1], [B2]) and/or Baker domains ([F]), also called “parabolic domains at ∞”.

We concentrate on the class of entire transcendental maps of finite type, that is

$$S = \{f : \mathbb{C} \to \mathbb{C},\ f \text{ trans. entire with only finitely many critical and asymptotic values}\}.$$

Dynamically, entire maps of finite type share some of the properties of polynomials since their Fatou sets cannot include wandering or Baker domains, nor Herman rings ([EL2, GK]).

Observe that the family of functions $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ belongs to S. The function $F_{\lambda,m}$ has two critical values at 0 and at $\lambda(-m)^m exp(-m)$, since the critical points are located at $z = 0$ and $z = -m$. It has also an asymptotic value at $v = 0$, since the function tends to 0 as z tends to ∞ along $\mathbb{R}^-$.

If $f \in S$, there exists a characterization of the Julia set ([EL1]), namely as the closure of the set of points whose orbits tend to ∞. Using the characterization above we can plot an approximation of $\mathcal{J}(F_{\lambda,m})$. Generally, orbits tend to ∞ in specific directions. In our case, if $\lim_{n\to\infty} |F^{\circ n}_{\lambda,m}(z)| = +\infty$, then we have $\lim_{n\to\infty} \mathrm{Re}(F^{\circ n}_{\lambda,m}(z)) = +\infty$. Thus, an approximation of the Julia set is given by the set of points whose orbit containts a point with real part greater than, say, 90. Observe that filled black regions are due to numerics, since the Julia set contains no open set.

In Figure 2, we display the Julia set of $F_{\lambda,m}$ for different values of λ and m. The immediate basin of attraction of $z = 0$ is shown [1] in blue, while the other components of $A_{\lambda,m}(0)\backslash A^*_{\lambda,m}(0)$ are shown in red. The components of the Fatou set different from $A_{\lambda,m}(0)$ are shown in orange. Points in the Julia set are shown in black. We show the dynamical plane of the function $F_{\lambda,2} = \lambda z^2 \exp(z)$, for three different values of λ and different ranges. As we proved in [FG] the basin of 0 contains an infinite number of horizontal strips, that extend to $+\infty$ as their real parts tend to $+\infty$. Between these strips we find the well known structures, named Cantor Bouquets which are invariant sets of curves governed by some symbolic dynamics. This kind of structures in the Julia set are typical for critically finite entire transcendental functions ([DT]). Also, as we change the parameter λ we observe that the relative position of these bands also changes, but not their width. Finally, we can see the existence of an unbounded region that extends to $-\infty$ contained in $A_{\lambda,m}(0)$.

In the zoom plates of Figure 2, range $(-1, 1) \times (-1, 1)$, we can see the dynamical plane near the origin. It seems that the immediate basin of attraction of $z = 0$ is a Jordan domain for $\lambda = -8$ and $\lambda = 6.9$.

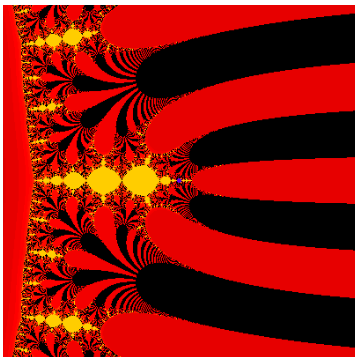
The orbit of the free critical point $z = -m$, determines in large measure the dynamics of $F_{\lambda,m}$. Indeed, the functions $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ are entire maps with a finite number of critical and asymptotic values, hence we know that if the orbit of $z = -m$ tends to ∞ no other Fatou components can exist besides those that belong to $A_{\lambda,m}(0)$. Hence the Fatou set must coincide with the basin of 0, i.e., $\mathcal{F}(F_{\lambda,m}) = A_{\lambda,m}(0)$. The set B_m is defined as

$$B_m = \{\lambda \in \mathbb{C} \,|\, F^{\circ n}_{\lambda,m}(-m) \nrightarrow \infty\}.$$

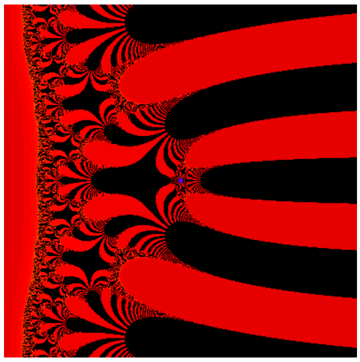
In each of these sets, we may also distinguish between two different behaviors: those

[1] Color plots are available in the online version of this paper. Otherwise, blue is darker than red and orange is light.

(a) $\lambda = -2.1$. Range $(-10, 10) \times (-10, 10)$.

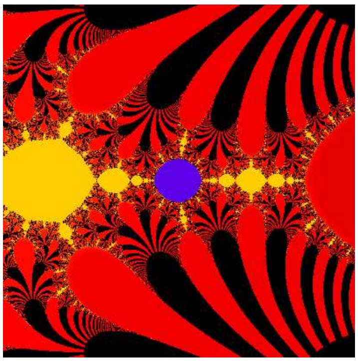


(b) $\lambda = -8$. Range $(-10, 10) \times (-10, 10)$.

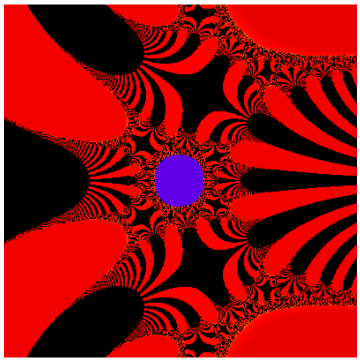


(c) $\lambda = 6.9$. Range $(-10, 10) \times (-10, 10)$.

(d) $\lambda = -2.1$. Range $(-1, 1) \times (-1, 1)$.



(e) $\lambda = -8$. Range $(-1, 1) \times (-1, 1)$.



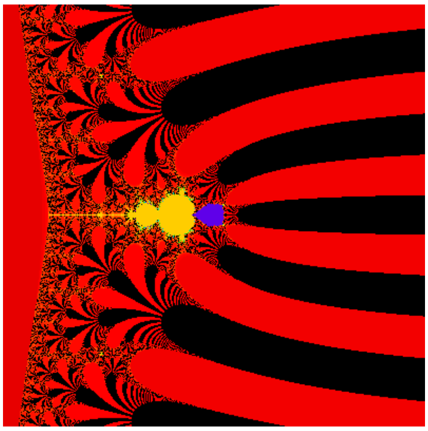
(f) $\lambda = 6.9$. Range $(-1, 1) \times (-1, 1)$.

Figure 2: The Julia set for $F_{\lambda,2}$.

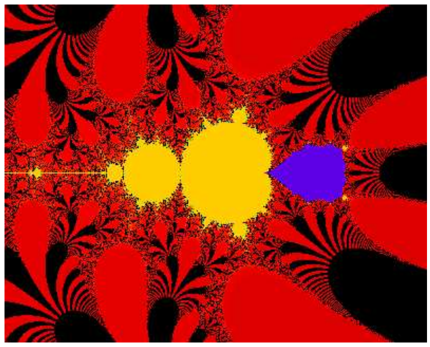
parameter values for which $-m \in A_{\lambda,m}(0)$ and those for which this does not occur. Let $\text{int}(B_m)$ denote the interior of B_m.

Definition. Let U be a connected component of $\text{int}(B_m)$. We say that U is a *capture zone* if for all λ in U we have that $\lim_{n\to+\infty} F^{\circ n}_{\lambda,m}(-m) = 0$, or in other words, $-m \in A_{\lambda,m}(0)$. We then say that the orbit of the critical point is captured by the basin of attraction of the superattracting fixed point $z = 0$.

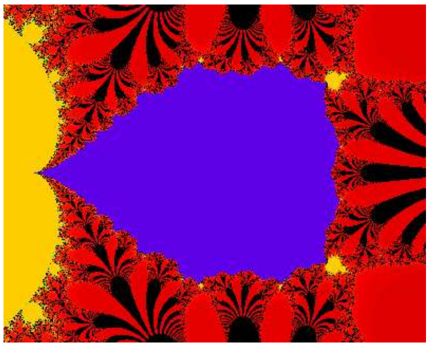
In Figure 3, we show a numerical approximation of the set B_2. The main capture zone is shown in blue, while other capture zones are shown in red. All other components of B_2 are shown in orange. The parameter values for which the orbit of the free critical point tends to ∞ are shown in black. In these sets we can see a countable quantity of horizontal strips. In Figure 3 (c) we can see the main capture zone $\mathcal{H}^0_m$.



(a) Range $(-25, 25) \times (-25, 25)$



(b) Range $(-15, 5) \times (-8, 8)$



(c) Range $(-3.1, 2.25) \times (-2.15, 2.15)$

Figure 3: Parameter plane for $F_{\lambda,2}$. Color codes are explained in the text.

4.1 Dynamical plane: proof of Proposition A

The first assertion of this theorem, i.e. that all connected components of the Fatou set are simply connected, is a general result for all functions in class S ([B]), which we have included here for completeness.

To see that the number of connected components of $A_{\lambda,m}(0)$ is either 1 or ∞, we observe that the basin of $z = 0$, $A_{\lambda,m}(0)$, consists of the immediate basin $A^*_{\lambda,m}(0)$ and all its preimages. For all connected components of $A_{\lambda,m}(0)$ other than $A^*_{\lambda,m}(0)$ there exists a number $i > 0$ such that $F^i_{\lambda,m}(U) \subset A^*_{\lambda,m}(0)$, where i is the smallest number with this property. Suppose that there exist a finite number of connected components, say $A^*_{\lambda,m}(0)\,,\,U_1\,,\,U_2\,,\,...\,,\,U_N$. By assumption, for each U_k there exist a number i_k such that $F^{i_k}_{\lambda,m}(U_k) \subset A^*_{\lambda,m}(0)$, for $1 \leq k \leq N$. Let i_l be the maximum of the indexes $i_1, \cdots, i_N$. Consider $z \in U_l$ such that is not exceptional; then, points in $F^{-1}_{\lambda,m}(z)$ belong to $A_{\lambda,m}(0)$, but not to $A^*_{\lambda,m}(0) \cup U_1 \cup ... \cup U_N$, which is a contradiction.

It remains to prove that all connected components of $A_{\lambda,m}(0)$ are unbounded except, maybe, $A^*_{\lambda,m}(0)$. To this end, suppose that U is a connected component of $A_{\lambda,m}(0)$ different from $A^*_{\lambda,m}(0)$, and let $i>0$ be the smallest number such that $F^i_{\lambda,m}(U) \subset A^*_{\lambda,m}(0)$. Let $z \in U$, and denote by γ a simple path in $A^*_{\lambda,m}(0)$ that joins $F^i_{\lambda,m}(z)$ and 0. The preimage of γ in U must include a path γ_1 that joins z and ∞, since 0 is an asymptotical value with no other finite preimage than itself. Thus we conclude that U is unbounded. This concludes the proof of Proposition A.

4.2 Parameter plane: proof of Theorem B

In this section we describe some properties of the capture zones $\mathcal{H}^n_m$. We are mainly interested in their topological properties. For clarity's sake we prove each of the statements in a different proposition.

Proposition 4.1. *The critical point $-m$ belongs to $A^*_{\lambda,m}(0)$ if and only if the critical value $F_{\lambda,m}(-m)$ belongs to $A^*_{\lambda,m}(0)$. Hence $\mathcal{H}^1_m = \emptyset$.*

Proof. Suppose that $F_{\lambda,m}(-m) \in A^*_{\lambda,m}(0)$. Let γ be a simple path in $A^*_{\lambda,m}(0)$ that joins $F_{\lambda,m}(-m)$ and 0. The set of preimages of γ must include a path γ_1 that joins $-\infty$ with $-m$, and also a path γ_2 that joins $-m$ and 0 (since $-m$ is a critical point and 0 is a fixed point and asymptotic value). Hence $\gamma_1 \cup \gamma_2 \subset A^*_{\lambda,m}(0)$ and so does $-m$. Conversely, if $-m \in A^*_{\lambda,m}(0)$ we have that $F_{\lambda,m}(-m) \in A^*_{\lambda,m}(0)$. □

We define $\rho = \min\{\frac{1}{e}, (\frac{e}{m})^m\}$, i.e., $\rho = 1/e$ for $m=2,3$ and $\rho = (\frac{e}{m})^m$ for $m \geq 4$. We also define $\rho' = (\frac{e}{m-1})^{m-1}$.

Proposition 4.2. *$D_\rho \subset \mathcal{H}^0_m \subset D_{\rho'}$.*

Proof. First we prove that $D_\rho = \{\lambda \in \mathbb{C}\,;\, |\lambda| < \rho\} \subset \mathcal{H}^0_m$. For $\lambda \in D_\rho$, we will prove that $F_{\lambda,m}(-m)$ lies in D_{ϵ_0} which we know belongs to $A^*_{\lambda,m}(0)$. In order to do so, we use that $\epsilon_0 \geq \min(1, (\frac{1}{|\lambda| e})^{1/(m-1)})$ (Lemma 2.2). If $\lambda \in D_\rho$, then $|\lambda| < \frac{1}{e}$, and hence $\epsilon_0 \geq 1$. The condition $\lambda \in D_\rho$ also implies that $|\lambda| < (\frac{e}{m})^m$. Hence

$$|F_{\lambda,m}(-m)| \;=\; |\lambda||(-m)^m e^{-m}| \;=\; |\lambda| \left(\frac{m}{e}\right)^m \;<\; 1 \leq \epsilon_0,$$

and $F_{\lambda,m}(-m)$ lies in $A^*_{\lambda,m}(0)$.

Second we prove that $\mathcal{H}^0_m \subset D_{\rho'}$. We will prove that $-m \notin A^*_{\lambda,m}(0)$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| > (\frac{e}{m-1})^{m-1}$. Let D be the disk centered at 0 of radius $m-1$. If we calculate the modulus of the image of its boundary, $\{|z| = m-1\}$, we obtain

$$|F_{\lambda,m}(z)| = |\lambda||z|^m e^{Re(z)} \geq |\lambda|(m-1)^m e^{-(m-1)} > m-1$$

where the inequality is obtained using $|\lambda| > (\frac{e}{m-1})^{m-1}$. This shows that $D \subset F_{\lambda,m}(D)$.

Let W be the component of $F_{\lambda,m}^{-1}(D)$ that contains the origin. It is clear that $\overline{W} \subset D$ and $\overline{A^*_{\lambda,m}(0)} \subset W$. Moreover, $F_{\lambda,m}$ is a proper function of degree m from W onto D, (see Figure 4). In the terminology of polynomial-like mappings, developed by Douady and Hubbard ([DH3]), the triple $(F_{\lambda,m}; W, D)$ is a polynomial-like mapping of degree m. By the Straightening Theorem ([DH3]), there exists a quasiconformal mapping, ϕ, that conjugates $F_{\lambda,m}$ to a polynomial P of degree m, on the set W. That is $(\phi^{-1} \circ F_{\lambda,m} \circ \phi)(z) = P(z)$ for all $z \in W$. Since $z = 0$ is superattracting for $F_{\lambda,m}$ and ϕ is a conjugacy, we have that $z = 0$ is superattracting for P. Hence, after perhaps a holomorphic change of variables, we may assume that $P(z) = z^m$.

Hence $A^*_{\lambda,m}(0) \subset D$. Since $-m \notin D$ we conclude that $\mathcal{H}^0_m$ is bounded.

□

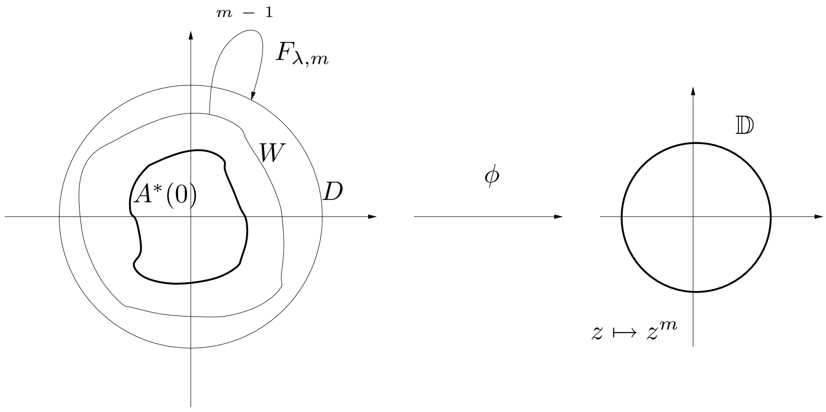


Figure 4: $F_{\lambda,m}$ is a polynomial-like mapping of degree m near the origin.

Proposition 4.3. *The main capture zone $\mathcal{H}^0_m$ is connected and simply connected.*

Proof. We prove that $\mathcal{H}^0_m$ is conformally a disk. Since $F_{\lambda,m}(z)$ has a superattracting fixed point at $z = 0$, we can use the Böttcher coordinate near the origin (see Section 3.1) to define a suitable biholomorphic map in the main capture zone. Using a suitable linear change of variables we obtain a new family of entire transcendental maps, so that near the superattracting fixed point $z = 0$, the functions can be written as $z^m + O(z^{m+1})$, and thus having a preferred Böttcher coordinate in this region (see Section 3.1). We consider,

$$L_{a,m}(z) = z^m e^{z/a}, a \in \mathbb{C} \setminus \{0\}, \text{ and } m \in \mathbb{N}, m \geq 2. \tag{11}$$

Under this map, the superattracting fixed point $z = 0$ is still at $z = 0$, and the free critical point (located at $z = -m$ for $F_{\lambda,m}$) is now at $c_{a,m} = -m\,a$ for $L_{a,m}$. We now define the following auxiliary set for the family of maps $L_{a,m}$ which is closely related to the main capture zone, more precisely

$$\hat{\mathcal{H}}_m^0 = \{a \in \mathbb{C}, \text{ such that } a^{m-1} \in \mathcal{H}_m^0\}$$

By construction, $a \in \hat{\mathcal{H}}_m^0 \to a^{m-1} \in \mathcal{H}_m^0$ is a $(m-1)$–fold branched covering.

We consider the following mapping

$$\begin{aligned} \Phi : \hat{\mathcal{H}}_m^0 &\to \mathbb{D} \\ a &\to \varphi_{a,m}(c_{a,m}) \end{aligned} \tag{12}$$

Where φ_a is the Böttcher coordinate defined in the immediate basin of attraction of $z = 0$ (Section 3.1 and Lemma 3.2). We claim that the map Φ is well defined and, in fact, is a conformal isomorphism which is tangent to $a \to \frac{-m}{e} a$ at the origin. If $a \in \hat{\mathcal{H}}_m^0 \setminus \{0\}$, the Böttcher map extends until the critical point $c_{a,m} = -ma$, and using Equation (10) we have that

$$\varphi_{a,m}(c_{a,m}) = (-ma) \exp\left[\sum_{n=0}^{\infty} \frac{L^{\circ n}(-ma)}{a\, m^{n+1}}\right]$$

We see inductively that $\frac{L^{\circ n}(-ma)}{a}$ is a holomorphic function of a^{m-1}. Indeed, using the definition of $L_{a,m}(z) = z^m e^{z/a}$, we have

$$\frac{L_{a,m}^{\circ 0}(-ma)}{a} = -m \qquad \text{and} \qquad \frac{L_{a,m}^{\circ 1}(-ma)}{a} = \left(\frac{-m}{e}\right)^m a^{m-1}.$$

Assuming then that $\frac{L^{\circ n}(-ma)}{a} = R(a^{m-1})$, where $R(w)$ is a holomorphic map on w, we see that

$$\frac{L_{a,m}^{\circ(n+1)}(-ma)}{a} = \frac{\left[L_{a,m}^{\circ n}(-ma)\right]^m \exp\left[\frac{L_{a,m}^{\circ n}(-ma)}{a}\right]}{a} = a^{m-1}[R(a^{m-1})]^m \exp[R(a^{m-1})]$$

proving thus that $\frac{L^{\circ n}(-ma)}{a}$ is a holomorphic function of a^{m-1}.

As $a \to 0$, a brief computation shows that $\varphi_{a,m}(c_{a,m}) = -\frac{m}{e}\, a\, \eta(a^{m-1})$, where $\eta(w)$ is a holomorphic mapping so that $\eta(0) = 1$. Hence the apparent singularity at $a = 0$ is removable. Since the correspondance $a \to \varphi_{a,m}(c_{m,a})$ (Equation 12) is well defined and holomorphic, it suffices to show that $\varphi_{a,m}(c_{a,m})$ is a proper map of degree one from $\mathcal{H}_m$ onto $\mathbb{D}$.

To this end, we first consider a boundary point $a_0 \in \partial\mathcal{H}_{0,m}$. Then, as noted earlier, the Böttcher mapping from the immediate basin $A^*_{a_0,m}(0)$ onto the unit disc has no critical points, and in fact is a conformal diffeomorphism. In particular, $\varphi_{a_0,m}$ can be defined as a single valued function on the disc of radius $1-\epsilon$, for any $\epsilon > 0$. This last property must be preserved under any small perturbation of a_0, and it follows that $|\varphi_{a,m}(-m\,a)| > 1-\epsilon$ for any $a \in \mathcal{H}_{0,m}$ sufficiently close to a_0. Thus Φ is a proper map from $\mathcal{H}_{0,m}$ onto $\mathbb{D}$. Since $\Phi^{-1}(0)$ is the single point 0, with $\Phi'(0) = -\frac{m}{e} \neq 0$, it follows that Φ is a conformal diffeomorphism.

We can define now the conformal mapping from $\mathcal{H}_m^0$ to $\mathbb{D}$ using the construction above. Since the conformal mapping $\Phi : \hat{\mathcal{H}}_m^0 \to \mathbb{D}$, writes as $\Phi(a) = \frac{-e}{m}\, a\, \eta(a^{m-1})$, it follows that Φ^{-1} sends a sector $S = \{z \in \mathbb{D}, 0 \le arg(z) \le \frac{2\pi}{m-1}\}$ into a sector $\Phi^{-1}(S) \subset \hat{\mathcal{H}}_m^0$ with an amplitude equal to $\frac{2\pi}{m-1}$. We can see that $S \cong \mathbb{D}$. Hence we obtain a conformal mapping from S to $\mathcal{H}_m^0$ defined as

$$\begin{aligned} \Lambda : & S \to \mathcal{H}_m^0 \\ & z \to \left[\Phi^{-1}(z)\right]^{m-1} \end{aligned} \tag{13}$$

□

Proposition 4.4. *For all $n, m \ge 2$, the connected components of $\mathcal{H}_m^n$ are unbounded.*

Proof. Let U be a connected component of a capture zone different from $\mathcal{H}_m^0$. We assume that U is bounded, then

$$\sup_{\lambda \in \partial U} |\lambda| = M_1 < +\infty.$$

Since $\lambda = 0$ belongs to $\mathcal{H}_m^0$, we observe that $0 \notin U$.

We claim that there exist $\epsilon_1(m) > 0$ such that for all $\lambda \in \partial U$, we have that $|F_{\lambda,m}^{\circ k}(-m)| \ge \epsilon_1(m)$ for all $k \ge 0$. To see this, we only need to prove that for all $\lambda \in \partial U$ we can find $\epsilon_1 > 0$ such that $D(0, \epsilon_1) \subset A_{\lambda,m}^*(0)$. For all $\lambda \in \mathbb{C}$ there exists $\epsilon_0 > 0$, depending on $|\lambda|$ and m, (Theorem 2.1) such that $D(0, \epsilon_0) \subset A_{\lambda,m}^*(0)$. We also know (see Lemma 2.2) that

$$\epsilon_0(|\lambda|, m) \ge \min\{1, \left(\frac{1}{|\lambda| e}\right)^{\frac{1}{m-1}}\}.$$

If λ belongs to ∂U, then $|\lambda| \le M_1$, and we have

$$\left(\frac{1}{|\lambda|\, e}\right)^{\frac{1}{m-1}} \ge \left(\frac{1}{M_1\, e}\right)^{\frac{1}{m-1}}.$$

Hence, we define $\epsilon_1 = \min\{1, \left(\frac{1}{M_1\, e}\right)^{\frac{1}{m-1}}\}$ and this proves the claim.

Let $\lambda_0 \in U$. Since U is a capture zone, by definition we have that $F_{\lambda_0,m}^{\circ k}(-m) \to 0$ as $k \to \infty$. Let $k_0 \ge 0$ be such that, for all $k \ge k_0$, we have $|F_{\lambda_0,m}^{\circ k}(-m)| < \epsilon_1/2$. We consider now the mapping,

$$\begin{array}{cccc} F_{\lambda,m}^{\circ k_0}(-m) : & U & \to & \mathbb{C} \\ & \lambda & \to & F_{\lambda,m}^{\circ k_0}(-m) \end{array}$$

On the one hand, this is a holomorphic function of λ. On the other hand, since $0 \notin U$, we have that $F_{\lambda,m}^{\circ k_0}(-m) \ne 0$ for all $\lambda \in U$ (the only preimage of $z = 0$ under $F_{\lambda,m}(z)$ is $z = 0$). If we apply the minimum principle to $F_{\lambda,m}^{\circ k_0}(-m)$, we have

$$\frac{\epsilon_1}{2} \ge |F_{\lambda_0,m}^{\circ k_0}(-m)| \ge \inf_{\lambda \in U} |F_{\lambda,m}^{\circ k_0}(-m)| = \inf_{\lambda \in \partial U} |F_{\lambda,m}^{\circ k_0}(-m)| \ge \epsilon_1$$

obtaining thus a contradiction.

□

Proposition 4.5. *For all $n, m \geq 2$, the connected components of $\mathcal{H}_m^n$ are simply connected.*

Proof. The proof uses a surgery construction (see Section 3.3 for preliminaries on this technique). Let U be a connected component of $\mathcal{H}_m^n$ where $m, n \geq 2$. We consider the following mapping

$$\begin{array}{rccl} \Phi_U: & U & \to & \mathbb{D} \setminus \{0\} \\ & \lambda & \to & \varphi_\lambda(F_{\lambda,m}^{\circ k+1}(-m)) \end{array}$$

where φ_λ denotes the Bötcher coordinate near the origin. As in the previous proposition the map Φ_U is a proper mapping and we will prove that it is a local homeomorphism.

Let $\lambda_0 \in U$ and $z_0 = \Phi_U(\lambda_0)$. The idea of this surgery construction is the following: for z near z_0 we can build a map $F_{\lambda(z),m}$ such that $F_{\lambda(z),m}^{k+1}(-m)$ has Böttcher coordinate z. We denote by W_{λ_0} the connected component of $A_{\lambda_0,m}(0)$ containing $F_{\lambda_0,m}^{\circ n}(-m)$, preimage of $A_{\lambda_0,m}^*(0)$. Let C_{λ_0} be an small open neighborhood of $F_{\lambda_0,m}^{\circ n+1}(-m)$ contained in $A_{\lambda_0,m}^*(0)$, and $B_{\lambda_0} \subset W_{\lambda_0}$ be the preimage of C_{λ_0} containing $F_{\lambda_0,m}^{\circ n}(-m)$.

For any $0 < \epsilon < \min\{|z_0|, 1 - |z_0|\}$ and any $z \in D(z_0, \epsilon)$, we choose a diffeomorphism $\delta_z : B_{\lambda_0} \to C_{\lambda_0}$ with the following properties:

- $\delta_{z_0} = F_{\lambda_0,m}$;
- δ_z coincides with $F_{\lambda_0,m}$ in a neighborhood of ∂B_{λ_0} for any z;
- $\delta_z(F_{\lambda_0,m}^{\circ k}(-m)) = \varphi_{\lambda_0}^{-1}(z)$.

We consider, for any $z \in D(z_0, \epsilon)$, the following mapping $G_z : \mathbb{C} \to \mathbb{C}$:

$$G_z(x) = \begin{cases} \delta_z(x) & \text{if } x \in B_{\lambda_0} \\ F_{\lambda_0,m}(x) & \text{if } x \notin B_{\lambda_0} \end{cases}$$

We proceed to construct an invariant almost complex structure, σ_z, with bounded dilatation ratio. Let σ_0 be the standard complex structure of $\mathbb{C}$. We define a new almost complex structure σ_z in $\mathbb{C}$.

$$\sigma_z := \begin{cases} (\delta_z)^* \sigma_0 & \text{on } B_{\lambda_0} \\ (F_{\lambda_0,m}^n)^* \sigma & \text{on } F_{\lambda_0,m}^{-n}(B_{\lambda_0}) \text{ for all } n \geq 1 \\ \sigma_0 & \text{on } \mathbb{C} \setminus \bigcup_{n \geq 1} F_{\lambda_0,m}^{-n}(B_{\lambda_0}) \end{cases} .$$

By construction σ is G_z-invariant, i.e., $(G_z)^* \sigma = \sigma$, and it has bounded distortion since δ_z is a diffeomorphism and F_{λ_0} is holomorphic. If we apply the Measurable Riemann Mapping Theorem (see Section 3.3 and Remark 3.6) we obtain a quasiconformal map $\tau_z : \mathbb{C} \to \mathbb{C}$ such that τ_z integrates the complex structure σ_z, i.e., $(\tau_z)^* \sigma = \sigma_0$, normalized so that $\tau(0) = 0$

and $\tau(-m) = -m$. Finally, we define $R_z = \tau_z \circ G_z \circ \tau_z^{-1}$, which is analytic, hence an entire function.

We claim that this resulting mapping is $R_z(x) = \lambda x^m \exp(x)$, for some λ. Indeed, the map $R_z : \mathbb{C} \to \mathbb{C}$ is an entire map (∞ is an essential singularity) with a superattracting fixed point at the origin. Moreover, R_z has a critical point at $z = -m$. Thus $R_z(x) = \nu x^m \exp(h_1(x))$.

It is easy to show that the growth order of $F_{\lambda,m}$ is equal to 1, hence R_z has the same growth order. We know that the composition of a function of finite growth order by a quasiconformal function can only change the growth order by a real factor ([G]). We can conclude that R_z has finite growth order, hence $h_1(x)$ is a polynomial of degree d. Then there are d directions where $Re(h_1(x)) \to +\infty$ and $Im(h_1(x))$ is bounded for $x \to \infty$, separated by d directions where $Re(h_1(x)) \to -\infty$ and $Im(h_1(x))$ is bounded. Thus there are d directions where $R_z \to \infty$ separated by d directions where $R_z \to 0$. This behavior is invariant under topological conjugation. Since $F_{\lambda_0,m}$ has only one direction (along the positive real axis) where $F_{\lambda_0,m} \to \infty$ and one (the negative real axis) where $F_{\lambda_0,m} \to 0$, we conclude that $d = 1$ and $R_z(x) = \nu x^m \exp(a_0 + a_1 x)$. If we use that $-m$ is a critical point, then a_1 must be equal to 1. Finally, if we define $\lambda = \nu \exp(a_0)$, we can conclude that $R_z(x) = \lambda x^m \exp(x)$.

By construction, τ_{z_0} is the identity for $z = z_0$, then there exists a continuous function $z \in D(z_0, \epsilon) \mapsto \lambda(z) \in U$ such that

$$\lambda(z_0) = z_0 \text{ and } F_{\lambda(z),m} = \tau_z \circ G_z \circ \tau_z^{-1}$$

Moreover, τ_z is holomorphic on $A^*_{\lambda_0,m}(0)$ conjugating $F_{\lambda_0,m}$ and $F_{\lambda(z),m}$. Hence, observing the following commutative diagram

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "z^2"] \arrow[d, "\varphi_{\lambda_0}"'] & \mathbb{D} \arrow[d, "\varphi_{\lambda_0}"] \\
A^*_{\lambda_0,m}(0) \arrow[r, "F_{\lambda_0,m}"] & A^*_{\lambda_0,m}(0) \\
A^*_{\lambda(z),m}(0) \arrow[r, "F_{\lambda(z),m}"] \arrow[u, "\tau_z"] & A^*_{\lambda(z),m}(0) \arrow[u, "\tau_z"']
\end{tikzcd}$$

we have that $\varphi_{\lambda(z)} = \varphi_{\lambda_0} \circ \tau_z^{-1}$ is the Böttcher coordinate of $A^*_{\lambda(z),m}(0)$. Finally we conclude that

$$\Phi_U(\lambda(z)) = \varphi_{\lambda(z)}(F^{\circ n+1}_{\lambda(z),m}(-m)) = z$$

since $F^{\circ n+1}_{\lambda(z),m}(-m) = \tau_z \circ G_z^{\circ n+1} \circ \tau_z^{-1}(-m) = \tau_z \circ G_z^{\circ n+1}(-m) = \tau_z \circ G_z(F^{\circ n}_{\lambda_0,m}(-m)) = \tau_z \circ \varphi^{-1}_{\lambda_0}(z) = \tau_z \circ \tau_z^{-1} \circ \varphi^{-1}_{\lambda(z)}(z) = \varphi^{-1}_{\lambda(z)}(z)$.

□

4.3 Parameter plane: proof of Theorem C

Proposition 4.6. *If $\lambda \in \mathcal{H}_m^0$ then $A_{\lambda,m}(0) = A^*_{\lambda,m}(0)$. Otherwise if $\lambda \notin \mathcal{H}_m^0$ then $A_{\lambda,m}(0)$ has infinitely many connected components.*

Proof. Let $\lambda \in \mathcal{H}_m^0$. As in Proposition 4.1, let γ be a simple path in $A^*_{\lambda,m}(0)$ that joins $F_{\lambda,m}(-m)$ and 0. The preimage of γ must include a path $\tilde{\gamma}$ contained in $A^*_{\lambda,m}(0)$ that joins $-\infty$ with 0 passing through $-m$ ($\tilde{\gamma}$ maps 2-1 to γ). Since $H_{|\lambda|,m}$ intersects $\tilde{\gamma}$, it follows that $H_{|\lambda|,m} \subset A^*_{\lambda,m}(0)$. We recall that $H_{|\lambda|,m}$ is a preimage of a small disk of radius ϵ_0 (see Section 2).

All preimages of $\tilde{\gamma}$, are contained in $A^*_{\lambda,m}(0)$ as well, since they all intersect $H_{|\lambda|,m}$. In fact, we have that $A_{\lambda,m}(0) = A^*_{\lambda,m}(0)$ since any preimage of D_{ϵ_0} must contain points of $H_{|\lambda|,m}$. Hence $A_{\lambda,m}(0)$ has a unique connected component.

Now assume $\lambda \notin \mathcal{H}_m^0$. From Proposition A(b) we have that $A_{\lambda,m}(0)$ has either one or infinitely many connected components. If we suppose that $A_{\lambda,m}(0)$ has only one connected component, then $A_{\lambda,m}(0)$ is a completely invariant component of the Fatou set. Then, all the critical values of $F_{\lambda,m}$ are in $A_{\lambda,m}(0)$ (See [B2]), and hence we conclude that $-m$ belongs to $A_{\lambda,m}(0)$. However, this is impossible if $\lambda \notin \mathcal{H}_m^0$.

□

Proposition 4.7. *If $\lambda \in \mathcal{H}_m^0$, the boundary of $A^*_{\lambda,m}(0)$ (which is equal to the Julia set) is a Cantor bouquet and it is not locally connected.*

Proof. Using the proposition above, if λ belongs to the main capture zone, $\mathcal{H}_m^0$, we have that $A^*_{\lambda,m}(0) = A_{\lambda,m}(0)$. Hence, the Fatou set contains a totally invariant component. In fact, from [BD], it follows that the Julia set has an uncountable number of connected components and it is not locally connected at any point.

Using standard techniques analogous to [DT] one can show that the Julia set contains a Cantor Bouquet tending to ∞ in the direction of the positive real axis. Indeed, it is sufficient to construct a hyperbolic exponential tract on which $F_{\lambda,m}$ has asymptotic direction θ^*. To this end, let B_r be an open disk containing $F_{\lambda,m}(-m)$. The preimage of this set is an open set similar to $H_{|\lambda|,m}$ (see Section 2). Let D be the complement of this set. We have that $F_{\lambda,m}$ maps D onto the exterior of B_r, then D is an exponential tract for $F_{\lambda,m}$.

We may choose the negative real axis to define the fundamental domains in D. More precisely, we can find the preimage of the negative real axis under the function $F_{\lambda,m}$. Hereafter, we denote by $Arg(.) \in (-\pi, \pi]$ the principal argument. Using the definition of $F_{\lambda,m}$ it is easy to see that

$$Arg(F_{\lambda,m}(z)) = Arg(\lambda) + mArg(z) + Im(z) \qquad \text{mod } (2\pi).$$

Finding the preimages of $\mathbb{R}^-$ is equivalent to solving

$$Arg(F_{\lambda,m}(z)) = \pi.$$

The equation above is equivalent to

$$Arg(\lambda) + m\alpha + rsin(\alpha) = (2k+1)\pi \qquad k \in \mathbb{Z},$$

where $r = |z|$ and $\alpha = Arg(z)$. Hence, we obtain

$$r = \rho(\alpha) = \frac{(2k+1)\pi - m\alpha - Arg(\lambda)}{sin(\alpha)} \qquad \alpha \in (-\pi, \pi).$$

We denote each of these curves by $\sigma_k = \sigma_k(\lambda, m)$, where the possible values of the argument depend on k. As their real parts tend to $+\infty$, the σ_k's are asymptotic to the lines $Im(z) = (2k+1)\pi - Arg(\lambda)$.

Since the curves σ_k for $k \in \mathbb{Z}$ are mapped by $F_{\lambda,m}$ onto the negative real axis, it follows that D has asymptotic direction $\theta^* = 0$. Furthermore, since $F_{\lambda,m}(z) = \lambda z^m \exp(z)$, one may check readily that D is a hyperbolic exponential tract. $\square$

Before proving assertion (c) of Theorem C we prove the following auxiliary lemma.

Lemma 4.8. *If $|\lambda| > (\frac{e}{m-1})^{m-1}$, then the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle.*

Proof. Let $\lambda \notin \mathcal{H}^0_m$ be such that $|\lambda| > (\frac{e}{m-1})^{m-1}$. By using the same arguments of Proposition 4.2 we have that $F_{\lambda,m}$ is a polynomial-like of degree m near the origin. From this construction we obtain that $\partial A^*_{\lambda,m}(0) = \phi(\mathbb{T})$, and the lemma follows.

$\square$

Remark 4.9. The reason to ask for $|\lambda| > (\frac{e}{m-1})^{m-1}$ as a condition is as follows. We want to find a value $K > 0$ such that if $|z| = K$ then $|F_{\lambda,m}(z)| > K$. This condition is equivalent to

$$|F_{\lambda,m}(z)| \geq |\lambda||z|^m e^{-|z|} = |\lambda|(K)^m e^{-K} > K$$

or equivalently

$$|\lambda| > K^{1-m}e^K.$$

We want to use this argument for the largest possible region of values of λ. Hence, we choose $K > 0$, such that $K^{1-m}e^K$ is minimum. This minimum value is reached exactly at $K = m-1$.

Proposition 4.10. *Let $\mathcal{U}_m$ be the unbounded connected component of $\mathbb{C} \setminus \overline{\mathcal{H}^0_m}$. If $\lambda \in \mathcal{U}_m$, then the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle.*

Proof. Let $\mathcal{U}_m$ be the unbounded component of $\mathbb{C} \setminus \overline{\mathcal{H}^0_m}$. Since $\mathcal{U}_m$ is unbounded let $\lambda_0 \in \mathcal{U}_m$ be such that $\partial A^*_{\lambda_0,m}(0)$ is a quasicircle (see Lemma 4.8), and hence $A^*_{\lambda_0,m}(0)$ is a quasidisk.

On the other hand, since $\mathcal{U}_m$ is an open and simply connected set, let $\psi : \mathbb{D} \to \mathcal{U}_m$ be the Riemann mapping such that $\psi(0) = \lambda_0$.

We claim that for all $\lambda \in \mathcal{U}_m$, the Böttcher mapping φ_λ conjugating $F_{\lambda,m}$ to $z \to \lambda z^m$ extends to the whole immediate basin of attraction $A^*_{\lambda,m}(0)$ (see Section 3.1). To see the claim we only need to observe that, when $\lambda \in U_m$ the critical point $-m$ does not belong to $A^*_{\lambda,m}(0)$, hence no other critical point than $z = 0$ belongs to $A^*_{\lambda,m}(0)$. It follows that for all $\lambda \in \mathcal{U}_m$, the Böttcher coordinate

$$\varphi_\lambda : A^*_{\lambda,m}(0) \to \mathbb{D},$$

is a conformal mapping.

We can define now a holomorphic motion of $A^*_{\lambda_0,m}(0)$ (see Section 3.2). We use as a main ingredients the Böttcher map, φ_λ, and the conformal Riemann mapping ψ. More precisely, we consider the following map

$$\begin{array}{rlll} \Phi: & \mathbb{D}\times A^*_{\lambda_0,m}(0) & \to & \mathbb{D}\times\mathbb{C} \\ & (c,z) & \to & (c,\Phi_c(z)) = (c,\Phi^z(c)) = (c,\varphi^{-1}_{\psi(c)}\circ\varphi_{\lambda_0}(z)) \end{array} \tag{14}$$

We can check that Φ is a holomorphic motion. By construction, we have that $\Phi_0(z) = \varphi^{-1}_{\psi(0)}\circ\varphi_{\lambda_0}(z) = z$. If we fix the parameter c we must see that the map $\Phi_c(z)$ is injective. This is immediate, since the the Bötcher mapping φ_λ is conformal. Finally, if we fix a point $z\in A^*_{\lambda_0,m}(0)$ we must see that $\Phi^z:\mathbb{D}\to\mathbb{C}$ is a holomorphic map. In this case the map Φ^z is a composition of holomorphic maps, since the Böttcher map depends analytically on parameters (see Figure 5).

Geometrically, if we fix $\lambda\in\mathcal{U}_m$, the map $z\to\Phi_{\psi^{-1}(\lambda)}(z)$ sends points in $A^*_{\lambda_0,m}(0)$ to points in $A^*_{\lambda,m}(0)$ according to the Böttcher coordinates.

Finally, we apply Lemma (3.4) to the holomorphic motion Φ, which roughly speaking, says that a holomorphic motion of a quasidisk is also a quasidisk.

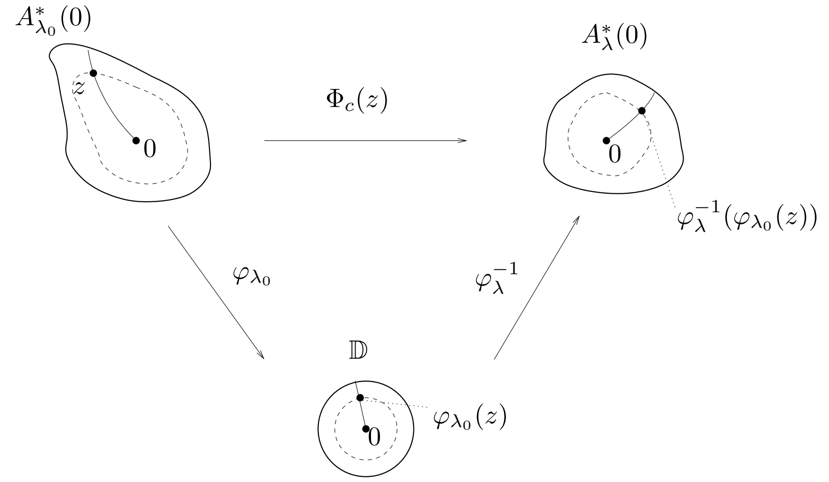


Figure 5: Sketch of the Holomorphic motion $\Phi_c(z)$, where $\lambda=\psi(c)$. Geometrically, $\Phi_c(z)$ sends equipotentials and rays from $A^*_{\lambda_0,m}(0)$ to $A^*_{\lambda,m}(0)$ according to Bötcher coordinates.

The final assertion of Theorem C(c), follows directly from the fact that all the sets $\mathcal{H}^n_m$ are unbounded and hence belong to $\mathcal{U}_m$

□

Remark 4.11. Since Φ extends to a holomorphic motion $\hat{\Phi}$ of $\overline{A^*_{\lambda_0,m}(0)}$, we have that for all $c\in\mathbb{D}$, $\hat{\Phi}_c(z_1)\neq\hat{\Phi}_c(z_2)$ for all $z_1,z_2\in\overline{A^*_{\lambda_0,m}(0)}$. In other words, if we take z_1 and z_2 in the boundary of $A^*_{\lambda_0,m}(0)$, the property above proves that two internal rays never land at a common point.

4.4 Parameter plane: proof of Theorem D

From Theorem B, statements b) and c), we know that $\mathcal{H}_m^0$ is bounded, connected and simply connected. As we mentioned in the introduction, we conjecture that $\mathcal{H}_m^0$ is a topological disc. If this were the case then $\mathbb{C}\setminus\overline{\mathcal{H}_m^0}$ would consist of only one connected component which would be unbounded. But as long as this result it is not proven, the complement of $\overline{\mathcal{H}_m^0}$ might have other connected components different from the unbounded one, which we denote by $\mathcal{U}_m$. In Theorem D we study the topological relation between these sets.

Proof. of Theorem D. In this proof we use the monic family of functions $L_{a,m} = z^m \exp(z/a)$ (See Section 3.1 and Proposition 4.3). We recall that $L_{a,m}(z)$ is conformally conjugate to $F_{\lambda,m}(z)$. We introduce this new family of maps in order to obtain a preferred Böttcher coordinate near $z = 0$. We also recall that the free critical point for the family $L_{a,m}(z)$ is at the point $c_{a,m} = -ma$. We denoted by φ_a the Böttcher coordinate defined in the whole immediate basin of attraction of $z = 0$ (Lemma 3.2) and by $\Phi(a) = \varphi_a(c_{a,m})$ (Equation 12) the uniformization mapping of the main capture zone (Proposition 4.3).

We want to show that $\mathcal{H}_m^0$ and $\mathcal{U}_m$ have a common boundary. Since $\mathcal{U}_m$ is the unbounded component of $\mathbb{C}\setminus\overline{\mathcal{H}_m^0}$ we have that $\partial\mathcal{U}_m \subset \partial\mathcal{H}_m^0$. Now, we will prove that $\partial\mathcal{H}_m^0 \subset \partial\mathcal{U}_m$ and thus statement a) follows. In order to do this, we first observe that the rest of the capture zones $\mathcal{H}_m^n$, for $n \geq 2$, are contained in $\mathcal{U}_m$ since they are unbounded and disjoint from $\mathcal{H}_m^0$. Second, notice that for any point a_0 in $\partial\mathcal{H}_m^0$, the sequence of $\{L_{a,m}^n(c_{a,m})\}_{n\geq 0}$ is not a normal family in any neighborhood of a_0.

Third, we claim that any arbitrary neighborhood of any point in $\partial\mathcal{H}_m^0$ meets $\mathcal{H}_m^n$ for some $n \geq 2$. To see the claim, let a_0 be a point in $\partial\mathcal{H}_m^0$, let W be a neighborhood of a_0. We must show that $W \cap \mathcal{H}_m^n \neq \emptyset$ for some $n \geq 2$. We also consider $\alpha' = 1/2$ and $\beta_1', \cdots, \beta_m'$ be complex numbers such that $(\beta_i')^m = \alpha'$.

Set

$$K = \{a \in \mathcal{H}_m^0 \,|\, |\Phi(a)| > |\sqrt[m]{\alpha'}|\} \text{ and } P = \mathbb{C}\setminus\{a \in \mathcal{H}_m^0 \,|\, |\Phi(a)| \leq |\sqrt[m]{\alpha'}|\}.$$

By shrinking W, if necessary, we can assume that $W \subset P$. Define functions $\alpha(a) = \varphi_a^{-1}(\alpha')$ and $\beta_i(a) = \varphi_a^{-1}(\beta_i')$ for $i = 1, \cdots, m$. See Figure 6 for a sketch of the relevant objects of this construction. Notice, that by construction of φ_a, if $a \in \mathcal{H}_m^0 \setminus \{0\}$ then the forward orbit of the free critical point $c_{a,m} = -ma$ is contained in $\varphi_a^{-1}(\mathbb{D}_{|\Phi(a)|})$. In particular, if $a \in K$ and $L_{a,m}^n(c_{a,m}) = \alpha(a)$, then

$$L_{a,m}^{n-1}(c_{a,m}) \in \{\beta_1(a), \cdots, \beta_m(a)\}$$

Now, let x_{a_0} be a preimage of $\alpha(a_0)$, that is not equal to $\beta_i(a_0)$ for any $1 \leq i \leq m$, notice that $L_{a,m}^n$ is ∞ to 1. We cannot have $c_{a,m} = x_{a_0}$ because then $L_{a,m}^n(c_{a,m})$ would be normal in a neighborhood of a_0. From the implicit function Theorem, we know that there exist a holomorphic function $x(a)$ such that $L_{a,m}(x(a)) = \alpha(a)$ in some neighborhood of a_0, which we can suppose is W by shrinking it, if necessary. Again by shrinking W, we can suppose that $x(a) \neq \beta_i(a)$ for all $a \in W$, for $i = 1, \cdots, m$. By lack of normality, the iterates $L_{a,m}^n(c_{a,m})$ do not avoid $0, \infty$ and $x(a)$. So, there exist $a' \in W$ and $n' \geq 0$ such that

$$L_{a',m}^{n'}(c_{a',m}) = x(a').$$

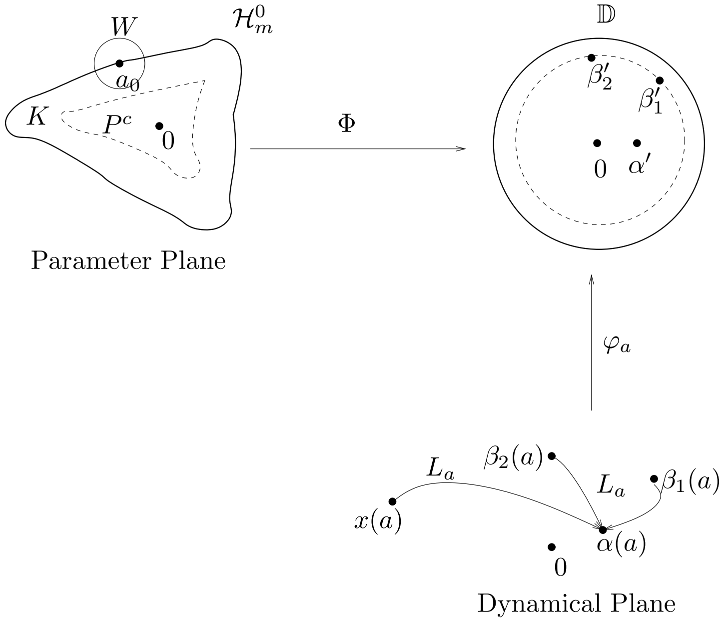


Figure 6: Sketch of the relevant objects in proof of Theorem D.

It follows that $a' \in \mathcal{H}_m^n$ for some $n \geq 0$. We finally claim that $n > 0$. If $n = 0$, then $a' \in K$, and $L_{a',m}^{n'+1}(c_{a',m}) = \alpha(a')$, this would mean that $L_{a',m}^{n'}(c_{a,m}) = \beta_i(a')$ for some i, a contradiction.

To prove the second statement of Theorem D, let $\mathcal{V}$ be a bounded connected component of $\mathbb{C}\backslash\overline{\mathcal{H}_m^0}$. Hence, we have that $\partial\mathcal{V} \subset \partial\mathcal{H}_m^0$ and $\partial\mathcal{V} \subset \partial\mathcal{U}_m$, since, by statement a), $\partial\mathcal{U}_m = \partial\mathcal{H}_m^0$. Then, $\mathcal{V}$ has a common boundary with $\mathcal{H}_m^0$ and $\mathcal{U}_m$.

□

5 A model for $F_{\lambda,m}$

For each natural value $m \in \mathbb{N}$, $m \geq 2$ and $\alpha, \beta \in \mathbb{R}$ we define the two-parameter family of maps

$$G_{\alpha,\beta,m}(z) = e^{i\alpha} z^m e^{\beta/2(z-1/z)}$$

It is easy to check that, $G_{\alpha,\beta,m}$ preserves the unit circle, S^1, and on this circle we have the following dynamical system

$$\widetilde{G}_{\alpha,\beta,m} : \theta \to \alpha + m\theta + \beta\sin(\theta) \qquad \text{mod } (2\pi), \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

When $\beta < 1$ and $m = 1$ this family of circle diffeomorphisms is known as the *standard family* or *Arnold family* and its parameter space contains the well known Arnold Tongues ([A]). When $m \geq 2$ the situation is very different because $\widetilde{G}_{\alpha,\beta,m}$ is an m to 1 map of $\mathbb{T}$ and hence not a circle diffeomorphism.

For each parameter value α and β the map $G_{\alpha,\beta,m}$ is a holomorphic function defined on the punctured plane, $\mathbb{C}^*$, with 0 and ∞ as essential singularities. We denote by $\mathcal{P}$ this class of functions. Maps of this type, are studied in [Ke], [Ko1], [Ko2] and [Mak] among others.

Let f be a holomorphic self-mapping of $\mathbb{C}^*$. The usual definitions of Fatou and Julia set apply for functions in class $\mathcal{P}$, although in this case, the Julia set can by characterized by the closure of the set of points whose orbits tend to 0 or to ∞ under iteration. Using the above characterization we can plot the Julia set of $G_{\alpha,\beta,m}$ for different values of α, β and m.

Sullivan's Theorem of nonwandering domains has been extended to the class $\mathcal{P} \cap S$ by many authors ([Ke], [Ko2]). Also for this kind of functions it is proved ([EL]) that they do not have Baker domains. Hence the classification of Fatou components is exactly the same as in the rational case.

The map $G_{\alpha,\beta,m}$ is of finite type, because it has only two critical points in $\mathbb{C}^*$. Indeed, if we compute $G'_{\alpha,\beta,m}$ we obtain

$$G'_{\alpha,\beta,m}(z) = \frac{1}{2} e^{i\alpha} z^{m-2} e^{\beta/2(z-1/z)} (\beta z^2 + 2mz + \beta),$$

and hence the two critical points $z_+(\beta)$ and $z_-(\beta)$ are given by,

$$z_\pm = \frac{-m \pm \sqrt{m^2 - \beta^2}}{\beta}.$$

In the case where α and β are real parameters we have that $G_{\alpha,\beta,m}$ is symmetric with respect to the unit circle which is also invariant. This condition is equivalent to

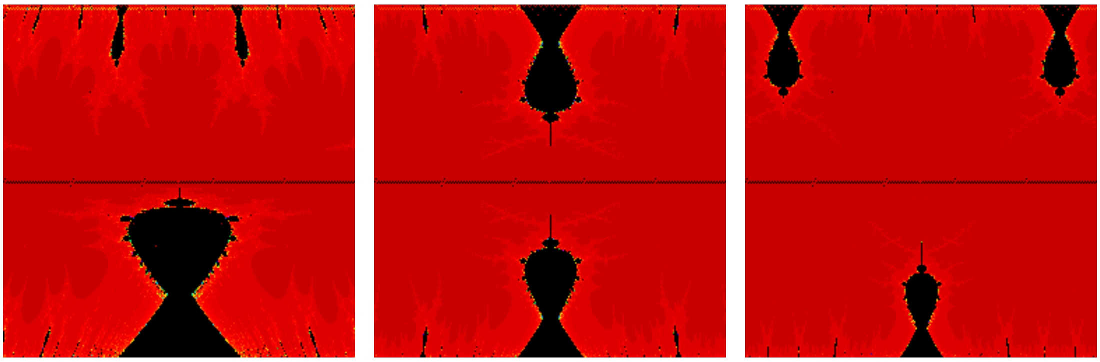
$$\tau \circ G_{\alpha,\beta,m} = G_{\alpha,\beta,m} \circ \tau$$

where $\tau(z) = 1/\overline{z}$. When $|\beta| < m$ the critical points $z_\pm$ have the same dynamical behavior since $\tau(z_-) = z_+$. Also, it is easy to check that z_+ belongs to $\mathbb{D}$ and hence $z_- \in \mathbb{C} \setminus \overline{\mathbb{D}}$.

In Figure 7 we display the parameter plane of $G_{\alpha,\beta,m}$ for $m = 2, 3$ and 4. We distinguish between two different behaviors of the free critical points $z_\pm$. Parameter values α and β for which the critical points tend to infinity or to zero are plotted in color, depending on the rate of escape. Parameter values α and β for which this does not occur are plotted in black. Black shapes that look like chess figures consist of parameter regions (shaped as Arnold tongues) where the attracting periodic orbit is contained in the unit circle and parameter regions (shaped as Mandelbrot set) where the attracting periodic orbit is disjoint from the unit circle. An exhaustive analysis of these Arnold tongues can be found in [MR].

In Figure 8 we display the dynamical plane of $G_{\alpha,\beta,m}$ for $m = 2$ and different values of α and β. Points tending to $z = 0$ and $z = \infty$ are shown in color, depending on the rate of escape, while points for which this not occur are shown in black. We also plot the unit circle in blue.

The following is the main idea of our surgery construction. First, we consider two real parameters α and β such that $\widetilde{G}_{\alpha,\beta,m}$ is quasisymmetrically conjugate to $\theta \mapsto m\theta$ on the unit circle. Under this conditions we can change the behavior of $G_{\alpha,\beta,m}$ on the unit disk. More precisely, we quasiconformally paste the superattracting behavior of $z \mapsto z^m$ inside the unit disk. The corresponding map acts like $G_{\alpha,\beta,m}$ outside on the complement of $\overline{\mathbb{D}}$ and acts like



(a) $m = 2$. Range $(-2,2) \times (-2,2)$.

(b) $m = 3$. Range $(-3,3) \times (-3,3)$.

(c) $m = 4$. Range $(-4,4) \times (-4,4)$.

Figure 7: Parameter planes of $G_{\alpha,\beta,m}$ for $m = 2,3$ and $m = 4$.

(a) $\alpha = 0.0$ and $\beta = -1.65$.

(b) $\alpha = 0$ and $\beta = 0.5$.

(c) $\alpha = 0.71$ and $\beta = 1.49$.

Figure 8: Dynamical plane of $G_{\alpha,\beta,m}$ for $m = 2$. Range $(-2,2) \times (-2,2)$.

$z \mapsto z^m$ on $\overline{\mathbb{D}}$. Second, applying the Measurable Riemann Mapping Theorem ([Ah, LV]) we can obtain a holomorphic mapping with this dynamical behavior, and finally we will prove that this map is precisely $F_{\lambda,m}(z) = \lambda z^m \exp z$ for some parameter λ. We obtain thus that $F_{\lambda,m}$ is quasiconformally conjugate on the complement of $A^*(0)$ to $G_{\alpha,\beta,m}$ on the complement of the closed unit disc.

5.1 The connection: Proof of Theorem E

Before proving Theorem E we can prove that $\mathcal{W}_m$ contains an open set of parameters. We recall that $\mathcal{W}_m$ is given by,

$$\mathcal{W}_m = \{\alpha, \beta \,|\, \widetilde{G}_{\alpha,\beta,m} \text{ is quasisymmetrically conjugate to } \theta \mapsto m\theta\}$$

Lemma 5.1. $\{(\alpha,\beta) \in \mathbb{R}^2 \,|\, |\beta| < m-1\} \subset \mathcal{W}_m$.

Proof. From Theorem 3.7 we can prove that $\widetilde{G}_{\alpha,\beta,m}$ is quasisymmetrically conjugate to $\theta \mapsto m\theta$ if we are able to prove that $\widetilde{G}_{\alpha,\beta,m}$ is an expanding map. In order to do so, a sufficient condition is to impose that $\min\{|\widetilde{G}'_{\alpha,\beta,m}(\theta)|, \theta \in \mathbb{T}\} > 1$. From the definition of $\widetilde{G}_{\alpha,\beta,m}(\theta) = \theta \mapsto \alpha + m\theta + \beta \sin\theta$ we have that

$$\widetilde{G}'_{\alpha,\beta,m}(\theta) \; = \; m + \beta \cos\theta.$$

Hence, it is easy to see that when $|\beta| < m-1$ we obtain that $\min\{|\widetilde{G}'_{\alpha,\beta,m}(\theta)|, \theta \in \mathbb{T}\} > 1$. ☐

Proof. of Theorem E.

Let α and β in $\mathcal{W}_m$. Let $h = h_{\alpha,\beta,m}$ be the quasisymmetric conjugacy, defined on the unit circle, such that $\widetilde{G}_{\alpha,\beta,m} = h^{-1} \circ g \circ h$, where $g(\theta) = m\theta$. Consider $\widehat{H} = \widehat{H}_{\alpha,\beta,m} : \mathbb{D} \to \mathbb{D}$ be the Douady-Earle quasiconformal extension of h such that $\widehat{H}(0) = 0$.

We now define a new function $R = R_{\alpha,\beta,m} : \mathbb{C} \to \mathbb{C}$ as follows:

$$R(z) \; := \; \begin{cases} G_{\alpha,\beta,m}(z) & z \notin \mathbb{D} \\ \widehat{H}^{-1}((\widehat{H}(z))^m) & z \in \mathbb{D} \end{cases}$$

This map is equal to $G_{\alpha,\beta,m}$ outside $\mathbb{D}$ and it has the desired superattracting dynamics in $\mathbb{D}$, but is not holomorphic on $\overline{\mathbb{D}}$. We proceed to construct an invariant almost complex structure, $\sigma = \sigma_{\alpha,\beta,m}$, with bounded dilatation ratio. Let σ_0 be the standard complex structure of $\mathbb{C}$. We define a new almost complex structure σ in $\mathbb{C}$.

$$\sigma \; := \; \begin{cases} (\widehat{H})^*\sigma_0 & \text{on } \mathbb{D} \\ (R^n)^*\sigma & \text{on } R^{-n}(\mathbb{D}) \text{ for all } n \geq 1 \\ \sigma_0 & \text{on } \mathbb{C} \setminus \bigcup_{n\geq 1} R^{-n}(\mathbb{D}) \end{cases} .$$

By construction σ is R-invariant, i.e., $(R)^*\sigma = \sigma$, and it has bounded distortion since $\widehat{H}$ is quasiconformal and R is holomorphic outside $\mathbb{D}$. If we apply the Measurable Riemann Mapping Theorem we obtain a quasiconformal map $\varphi = \varphi_{\alpha,\beta,m} : \mathbb{C} \to \mathbb{C}$ such that φ integrates the complex structure σ, i.e., $(\varphi)^*\sigma = \sigma_0$, normalized so that $\varphi(0) = 0$ and $\varphi(z_-) = -m$. Finally, we define $\tilde{R} = \tilde{R}_{\alpha,\beta,m} = \varphi \circ R \circ \varphi^{-1}$, which is analytic, hence an entire function. Our goal now is to show that there exist a complex value λ such that $\tilde{R}(z) = \lambda z^m \exp(z)$.

The map $\tilde{R} : \mathbb{C} \to \mathbb{C}$ is an entire map (∞ is an essential singularity) with a superattracting fixed point at the origin. Near the origin $\tilde{R}$ is conjugate to the map $z \mapsto z^m$. Moreover, $\tilde{R}$ has a critical point at $z = -m$, since the map R has one critical point at $z_- \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\varphi(z_-) = -m$. The other critical point of $G_{\alpha,\beta,m}$ is at z_+ and it has been erased by the quasiconformal surgery construction because it belonged to $\mathbb{D}$. Thus $\tilde{R}(z) = \nu z^m \exp(h_1(z))$. By using the same arguments as in Proposition 4.5 we can conclude that $\tilde{R}(z) = F_{\lambda,m}(z) = \lambda z^m \exp(z)$ for a suitable value of λ.

By construction, the boundary of $A^*_{\lambda,m}(0)$ is a quasicircle, since $A^*(0)$ is the quasiconformal image of the unit disk. Obtaining thus a value $\lambda \in \mathbb{C}$ such that $\partial A^*_{\lambda,m}(0)$ is a quasicircle. ☐

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