A mechanism for the fractalization of invariant curves in quasi-periodically forced 1-D maps

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Abstract

We focus on the continuation with respect to parameters of smooth invariant curves of quasi-periodically forced 1-D systems. In particular, we are interested in mechanisms leading to the destruction of the curve. One of these mechanisms is the so-called fractalization: the curve gets increasingly wrinkled until it stops being a smooth curve.

Here we show that this situation can appear when the Lyapunov exponent of a smooth non reducible curve (a curve whose linear normal behaviour cannot be reduced to constant coefficients) goes from a strictly negative value to zero. More concretely, using the Implicit Function Theorem (IFT) we show that an attracting curve can always be locally continued w.r.t. parameters inside its differentiability class, and that a zero Lyapunov exponent implies a failure of the IFT. In our scenario, the curve can only become fractal when the Lyapunov exponent vanishes. We illustrate these phenomena with some examples, including the quasi-periodically forced logistic map and an example based on the one used by G. Keller to prove the existence of Strange Non-chaotic Attractors.

Contents

1 Introduction

The continuation of invariant objects (fixed points, periodic orbits, invariant curves, higher dimensional tori, etc.) is an essential procedure to understand the geometrical structures that organise the phase space of a dynamical system. Here we will focus on the continuation of invariant curves of quasi-periodically forced one-dimensional dynamical systems,

$$\left.\begin{array}{rcl} \bar{x} & = & f_\mu(x,\theta), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\} \tag{1}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{T}^1$, $\mu \in \mathbb{R}$ is a parameter and f_μ is a smooth function. The value ω belongs to the set $(0,2\pi) \setminus 2\pi\mathbb{Q}$. To facilitate the reading we will abuse the language and refer to these numbers simply as irrationals (conversely, if $\omega \in 2\pi\mathbb{Q}$, we will call it rational). As ω is irrational, this system has neither fixed nor periodic points. Therefore, the simplest invariant objects are curves, that can be seen as the natural extension of the fixed or periodic points of f when f does not depend on θ.

Due to the rotation $\theta \mapsto \theta+\omega$, any orbit or invariant curve of (1) has zero as one of its two Lyapunov exponents. Therefore, we will ignore this zero exponent and we will focus on the remaining one that is related to the dynamics in the x direction. The curve is said to be attracting when this Lyapunov exponent is negative.

In this work we focus on the continuation w.r.t. parameters of a smooth and attracting invariant curve. More concretely, we are interested in the mechanisms that can lead to a destruction of the curve. One of these mechanisms is the so-called fractalization, that can be shortly descibed as a process in which the curve gets increasingly wrinkled until it stops being a smooth curve. There are numerical experiments in the literature claiming that the fractalization and the destruction of the curve take place when it is still attracting (for a survey, see [PNR01]). In this case the curve seems to be replaced by a Strange Non-chaotic Attractor (SNA): an attracting set with fractal dimension strictly larger than 1 and negative Lyapunov exponent.

In this paper we focus on a certain kind of failure for the continuation process of an attracting curve, that results in a fractalization phenomena when the Lyapunov exponent goes to zero. It is remarkable that, although the curve gets highly twisted, it keeps being a smooth curve as long as the Lyapunov exponent is strictly negative. We do not claim that this is the only scenario giving rise to fractalization (see Section 4.2), but we believe that some of the SNAs reported in the literature belong to the class considered here and, therefore, they are not "strange sets" but simply smooth (but highly wrinkled) curves.

More concretely, let us consider an attracting invariant curve whose linear normal behaviour is not reducible (see Definition 2.1). Then, we look at the continuation of this curve w.r.t. a parameter μ by applying the Implicit Function Theorem (IFT). It is well known that, when the spectrum of a suitable operator (3) does not contain 0, the IFT implies that the curve can be locally continued and, when 0 belongs to this spectrum, a bifurcation may take place. Here we prove that 0 does not belong to this spectrum as long as the curve is attracting, which implies that an attracting curve can always be locally continued inside its smoothness class. The fact that the curve is not reducible implies that the spectrum of the operator in the IFT does not contain eigenvalues. Therefore, when 0 enters in the spectrum we have a bifurcation that cannot be studied with standard methods (for instance, as 0 is not an eigenvalue we do not have centre manifold). We study this situation and we prove, in a concrete model, that the curve fractalizes for $\mu \to \mu^*$, but it keeps being a smooth curve as long as μ has not reached the critical value μ^*. The Lyapunov exponent of the curve is negative and goes to zero when $\mu \to \mu^*$. It is interesting to note that the numerical simulation seems to show the existence of a SNA when μ is close (but not equal) to μ^*, although we rigorously prove in our example that it is a C^∞ curve.

With this in mind, we give a close look to a well known example: the quasi-periodically forced logistic map. There we select a set of parameters for which it seems that this model has a SNA, and we try to numerically detect whether it is a smooth curve or not. The result is that, after a magnification of the order 10^{10} (using extended precision), it looks like a smooth curve. Of course, if one moves the parameters a bit such that the curve is more and more twisted, it becomes impossible to know numerically if it is still a curve or not. Our point is that, in this example, there is the same numerical evidence to claim that there is a SNA, than to claim the opposite.

We also prove that, for non-invertible quasi-periodically forced 1-D maps, repelling non reducible curves are not persistent under perturbations. This implies that one cannot expect to find them in a given system. This also implies that, when a non reducible attracting invariant curve becomes repelling, it should disappear.

In the proofs we take advantage of the low dimensionality of the system. In particular, we have been able to write normal forms for non reducible linear skew-products, including a normal form for the transition from reducibility to non reducibility. This allows, among other things, to prove that the dependence on parameters of the Lyapunov exponent is only C^0 at the point when the reducibility is lost. We illustrate this phenomenon in the quasi-periodically forced logistic map.

A preliminary version of these results can be found in [JT05].

1.1 The continuation of invariant curves

Let us assume that, for a given value of $\mu = \mu_0$, (1) has an invariant curve $x = u_{\mu_0}(\theta)$ with rotation number ω. For the moment being, we will assume that the curve is of class C^r, $r \geq 0$, but to speak about fractalization we will require more regularity than C^0. At this point we recall that, if a map of class C^r has a C^0 attracting invariant curve, the curve must be of class C^r (see [Sta97, HL05b]). Going back to the notation, and without loss of generality, we take $\mu_0 = 0$. Then, the invariant curve $u_0(\theta)$ satisfies the functional equation $F(u_0, 0) = 0$, where $F : C^r(\mathbb{T}^1, \mathbb{R}) \times \mathbb{R} \to C^r(\mathbb{T}^1, \mathbb{R})$ and, if $(u, \mu) \in C^r(\mathbb{T}^1, \mathbb{R}) \times \mathbb{R}$, we have

$$F(u, \mu)(\theta) = f_\mu(u(\theta), \theta) - u(\theta + \omega). \tag{2}$$

We are interested in the continuation of this curve w.r.t. the parameter μ, i.e., we look for a regular function $\mu \mapsto u_\mu$, defined for $|\mu|$ small enough, such that $F(u_\mu, \mu) = 0$.

We will work on the Banach space $C^r(\mathbb{T}^1, \mathbb{R})$, endowed with the standard C^r norm. To apply the Implicit Function Theorem (IFT) to (2) we have to check that F is differentiable and that $D_u F(u_0, 0)$ is an invertible bounded operator acting on $C^r(\mathbb{T}^1, \mathbb{R})$. The differentiability of F w.r.t. u follows from the smoothness of f and that $u(\theta) \mapsto u(\theta+\omega)$ is a linear bounded operator w.r.t. u. It is easy to check that, for any $(u,\mu) \in C^r(\mathbb{T}^1,\mathbb{R})\times\mathbb{R}$, and for any $v \in C^r(\mathbb{T}^1,\mathbb{R})$, we have that the function $D_uF(u,\mu)v \in C^r(\mathbb{T}^1,\mathbb{R})$ is given by

$$[D_uF(u,\mu)v](\theta) = D_xf_\mu(u(\theta),\theta)v(\theta) - v(\theta+\omega). \tag{3}$$

It is immediate to verify that $D_uF(u,\mu)$ is a bounded operator. Therefore, our main concern will be the existence of a bounded inverse for $D_uF(u_0,0)$ or, in other words, if zero belongs to the spectrum of $D_uF(u_0,0)$. One of the main results of this paper is that zero does not belong to the spectrum of $D_uF(u_0,0)$ if and only if the Lyapunov exponent of the invariant curve is negative. This implies that smooth and attracting invariant curves can always be locally continued inside the same smoothness class.

As we are dealing with infinite dimensional operators, the spectral values do not need to be eigenvalues. As we will see, this difference is very important because bifurcations due to an spectral value which is not an eigenvalue are completely different from bifurcations due to eigenvalues.

If zero is an eigenvalue, the possible bifurcations are the same ones as for the bifurcations of autonomous 1–D maps. The proof is based on showing that if zero is an eigenvalue, the linearization of the dynamics around the invariant curve can be reduced to constant coefficients (i.e., the invariant curve is reducible, see Section 2.1) which allows to use the standard normal form machinery (see, for instance, [BHTB90, BHJ+03]). If zero belongs to the spectrum of $D_uF(u_0,0)$ but it is not an eigenvalue, the situation is more complicated, and the standard normal form techniques cannot be used (for an example of this situation, see [BTW99]).

The paper is organised as follows. Section 2 focuses on general properties of lineal skew products (mainly reducibility, Lyapunov exponents and normal forms), Section 3 is devoted to the fractalization phenomena in affine systems and Section 4 is devoted to the examples and applications. We have also included some extra comments in Section 5.

2 Linear skew-products

If $x = u_0(\theta)$ is an invariant curve of class C^r, $r \geq 0$, its linearised normal behaviour is described by the following linear skew-product:

$$\left.\begin{array}{rcl} \bar{x} &=& a(\theta)x, \\ \bar{\theta} &=& \theta+\omega, \end{array}\right\} \tag{4}$$

where $a(\theta) = D_xf_0(u_0(\theta),\theta)$ is also of class C^r, $x \in \mathbb{R}$ and $\theta \in \mathbb{T}^1$. We will assume that the invariant curve is not degenerate, in the sense that the function $a(\theta)$ is not identically zero. The goal of this section is to derive several important properties of (4) that will enable us to connect the attracting character of the curve with its continuation w.r.t. parameters.

2.1 Reducibility

Definition 2.1 *The system (4) is called reducible iff there exists a (may be complex) change of variables $x = c(\theta)y$, continuous w.r.t. θ, such that (4) becomes*

$$\left.\begin{array}{rcl}\bar{y} & = & by, \\ \bar{\theta} & = & \theta+\omega,\end{array}\right\} \tag{5}$$

where b does not depend on θ.

Note that, if (4) is reducible with $b \neq 0$, the identity $b = a(\theta)c(\theta)/c(\theta+\omega)$ implies that $a(\theta)$ never vanishes. In particular, this implies that if a has zeros, the skew-product cannot be reducible. Moreover, if the transformation $x = c_1(\theta)y$ reduces (4) to (5) with a complex $b = b_1$, then the real change $x = |c_1(\theta)|y$ reduces (4) to $b = \mathrm{sign}\,(a)|b_1| \in \mathbb{R}$.

Proposition 2.1 *Given the skew-product (4), consider a linear skew-product*

$$\left.\begin{array}{rcl}\bar{y} & = & b(\theta)y, \\ \bar{\theta} & = & \theta+\omega,\end{array}\right\} \tag{6}$$

and assume that a and b are C^∞ functions. If there exists $\gamma > 0$ and $\tau \geq 1$ such that

$$|q\omega - 2\pi p| \geq \frac{\gamma}{|q|^\tau}, \qquad \textit{for all } (p,q) \in \mathbb{Z}\times(\mathbb{Z}\setminus\{0\}), \tag{7}$$

then there exists a C^∞ strictly positive function c such that the change $x = c(\theta)y$ transforms (4) into (6) if and only if the following two conditions are met:

1. $\dfrac{a(\theta)}{b(\theta)}$ *can be extended to a C^∞ strictly positive function for all θ,*
2. $$\int_0^{2\pi} \ln\frac{a(\theta)}{b(\theta)}\,d\theta = 0.$$

Proof: Let $c : \theta \in \mathbb{T}^1 \mapsto \mathbb{R}$ be a strictly positive function. The transformation $x = c(\theta)y$ brings (4) into the form

$$\bar{y} = a(\theta)\frac{c(\theta)}{c(\theta+\omega)}y.$$

Now, let us consider the equation

$$\frac{a(\theta)}{b(\theta)} = \frac{c(\theta+\omega)}{c(\theta)}, \qquad \text{for all} \quad \theta\in\mathbb{T}. \tag{8}$$

Assume that conditions 1. and 2. hold. Then, taking logarithms we obtain

$$\ln\frac{a(\theta)}{b(\theta)} = \ln c(\theta+\omega) - \ln c(\theta). \tag{9}$$

If we denote as α_k and c_k the Fourier coefficients of $\ln \frac{a(\theta)}{b(\theta)}$ and $\ln c(\theta)$, we find that c_0 is undefined (this is expected since the reduced transformation is defined modulus products by scalars). If $k \neq 0$ we have

$$c_k = \frac{\alpha_k}{\exp(\mathrm{i}\, k\omega) - 1}. \tag{10}$$

Using the Diophantine condition (7) and the smoothness of $\ln \frac{a(\theta)}{b(\theta)}$ (which implies a suitable decay on the values $|\alpha_k|$ when $|k|$ grows) we can also show a suitable decay on the values $|c_k|$, which shows the smoothness of $\ln c(\theta)$.

Now assume that there exists a C^∞ change $x = c(\theta)y$ that transforms (4) into (6). Then, condition 1. follows from (8) and condition 2. follows from (9). ■

Corollary 2.1 *Assume that ω satisfies the Diophantine condition (7) and that a is C^∞. Then, (4) is reducible if and only if a has no zeros.*

Proof: Assume first that (4) is reducible. If a has zeros, then the reduced system (5) must have $b = 0$, which in turn implies that $a \equiv 0$. This contradicts our assumption that a is not the null function.

Assume now that a has no zeros. Let us define the value

$$b = \operatorname{sign}(a) \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln|a(\theta)|\, d\theta\right] \neq 0,$$

where $\operatorname{sign}(a)$ denotes the sign of the function a, which is well defined since a does not have zeros. With this definition of b it is immediate to see that conditions 1 and 2 in Proposition 2.1 hold. Therefore, (4) is reducible to (6) with $b(\theta) \equiv b$. ■

Proposition 2.1 and Corollary 2.1 also hold if a and b are C^r functions, for r big enough. In this case, note that the effect of the small divisors in (10) does not allow to show that the reducing transformation $x = c(\theta)y$ is also C^r w.r.t. θ. The next result shows that this loss of differentiability for c is unavoidable.

Proposition 2.2 *Assume that ω is irrational. Then, there exist a strictly positive function $a \in C^r(\mathbb{T}^1, \mathbb{R})$ such that (4) is not reducible by means of a C^r transformation.*

Proof: Let $a \in C^r(\mathbb{T}^1, \mathbb{R})$ be a strictly positive function. Assume that there exists a transformation $x = c(\theta)y$ casting (4) into (5). Then, b and c satisfy the equation

$$\frac{c(\theta+\omega)}{c(\theta)} = \frac{a(\theta)}{b}$$

Taking logarithms and defining $\alpha = \ln a$, $\lambda = \ln b$ and $d = \ln c$, we have

$$d(\theta+\omega) - d(\theta) = \alpha(\theta) - \lambda$$

As the left hand side has zero average, λ has to be the average of α, namely

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} \ln a(\theta)\, d\theta.$$

To complete the proof, we will show that there exist functions $\beta \in C^r(\mathbb{T}^1, \mathbb{R})$, with zero average, for which there is no $d \in C^r(\mathbb{T}^1, \mathbb{R})$ such that $d(\theta + \omega) - d(\theta) = \beta(\theta)$.

Let us define $C_0^r = \{\varphi \in C^r(\mathbb{T}^1, \mathbb{R})$ such that $\int_0^{2\pi} \varphi(\theta)\, d\theta = 0\}$, and let us denote by T_ω the automorphism of C_0^r defined by $(T_\omega \varphi)(\theta) = \varphi(\theta + \omega)$. As T_ω is isometric, $\mathrm{Spec}\,(T_\omega) \subset \mathbb{S}^1$. Moreover, for all $k \in \mathbb{Z}$, $T_\omega(\exp(\mathrm{i}\, k\theta)) = \exp(\mathrm{i}\, k\omega)\exp(\mathrm{i}\, k\theta)$ which implies that, if $k \neq 0$, $\exp(\mathrm{i}\, k\omega)$ is an eigenvalue (the eigenfunction for $k = 0$ does not belong to C_0^r). Hence, $\mathrm{Spec}\,(T_\omega) = \mathbb{S}^1$.

As the spectral value 1 is not an eigenvalue, the range of the operator $T_\omega - \mathrm{Id}$ is not C_0^r. Hence, there exist functions $\beta \in C_0^r$ for which there is no $d \in C_0^r$ such that $d(\theta + \omega) - d(\theta) = \beta(\theta)$. ∎

In particular, this result implies that there exist strictly positive functions $a \in C^0$ for which (4) is not reducible to constant coefficients, since the reducing transformation cannot be continuous.

2.2 Normal forms and Lyapunov exponents

Proposition 2.1 can be used to derive a normal form for 1-D skew products.

Proposition 2.3 *Assume that ω satisfies the Diophantine condition (7) and that a is a C^∞ function with finitely many zeros, each with finite multiplicity. If n_0 is the total number of zeros –including multiplicities–, then there exists a unique trigonometric polynomial of degree n_0 such that the linear skew product (4) can be transformed into*

$$\left.\begin{array}{rcl} \bar{x} &=& p(\theta)x, \\ \bar{\theta} &=& \theta + \omega, \end{array}\right\}$$

The transformation also belongs to the C^∞ class.

Proof: Let q a non-zero polynomial of degree n_0 with the same zeros, including multiplicities, as a. Note that all the polynomials with these properties can be written as λq, for $\lambda \neq 0$. Note that q can be selected such that $q(\theta)a(\theta) \geq 0$, for all θ. Therefore, for any $\lambda > 0$, the quotient

$$\frac{a(\theta)}{\lambda q(\theta)}$$

can be extended to a strictly positive C^∞ function for all θ and, hence, condition 1 in Proposition 2.1 holds. Note that the value

$$\lambda = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log\frac{a(\theta)}{q(\theta)}\, d\theta\right],$$

is the only choice to satisfy condition 2 in Proposition 2.1. ∎

As the right-hand side of (4) is a linear function of x, the Lyapunov exponent of an orbit starting at (θ, x) only depends, in principle, of θ. This is the reason of the following definition.

Definition 2.2 *If $\theta \in \mathbb{T}^1$, we define the Lyapunov exponent of (4) at θ as*

$$\lambda(\theta) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| \prod_{j=0}^{n-1} a(\theta + j\omega) \right|. \tag{11}$$

We also define the Lyapunov exponent of the skew product (4) as

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)|\, d\theta.$$

If Λ is finite then the Birkhoff Ergodic Theorem says that, for Lebesgue-a.e. $\theta \in \mathbb{T}^1$, the $\limsup$ in (11) is in fact a lim and $\lambda(\theta) = \Lambda$. If $a(\theta)$ never vanishes, the $\limsup$ in (11) is again a lim and coincides with Λ but now for all $\theta \in \mathbb{T}^1$. In this last case, (11) converges uniformly. This follows from Proposition 4.1.13 in [KH95] using that irrational rotations on $\mathbb{T}^1$ are uniquely ergodic.

We have shown, in Proposition 2.1, that the zeroes of a are preserved by linear changes of variables so that they can be seen as an invariant of the cocycle. Now we will assume that a depends on a parameter μ, and we will focus on the regularity of the Lyapunov exponent Λ_μ w.r.t. μ. Roughly speaking, next result shows that Λ_μ depends smoothly on μ, except when the number of zeroes of a changes. In this last case, Λ_μ is only C^0.

Theorem 2.1 *Let us consider a one-parametric family of linear skew-products*

$$\left.\begin{array}{rcl} \bar{x} & = & a(\theta,\mu)x, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\} \tag{12}$$

where ω is Diophantine (see(7)) and μ belongs to an open subset of $\mathbb{R}$ and a is a C^∞ function of θ and μ. We assume that:

1. *For each μ, $a(\cdot,\mu)$ has finitely many zeros, each of them are simple except maybe one of multiplicity 2.*

   Let us call M the (open) set of values of μ for which all the zeros of $a(\cdot,\mu)$ are simple.

2. *If $a(\cdot,\mu)$ has a zero of multiplicity 2 at $\theta = \theta_0$ for $\mu = \mu_0$, then*

$$\frac{\partial a}{\partial \mu}(\theta_0,\mu_0) \neq 0.$$

Then, the Lyapunov exponent $\Lambda(\mu)$ of (12) is a continuous function of μ such that:

1. *Λ is C^∞ on M.*
2. *If $\mu_0 \notin M$, then*
   (a) *If the function "number of zeros of $a(\cdot,\mu)$" is increasing at μ_0, then*

$$\lim_{\mu\to\mu_0^-} \Lambda'(\mu) = -\infty, \ \textit{and} \ \lim_{\mu\to\mu_0^+} \Lambda'(\mu) \textit{ exists and is finite.}$$

(b) If the function "number of zeros of $a(\cdot,\mu)$" is decreasing at μ_0, then

$$\lim_{\mu\to\mu_0^-} \Lambda'(\mu) \text{ exists and is finite, and } \lim_{\mu\to\mu_0^+} \Lambda'(\mu) = +\infty.$$

Moreover, for $\mu \to \mu_0^-$ in (a) and for $\mu \to \mu_0^+$ in (b), we have the asymptotic expression

$$\Lambda(\mu) = \Lambda(\mu_0) + A\sqrt{|\mu-\mu_0|} + O(|\mu-\mu_0|), \tag{13}$$

where $A > 0$.

The proof is based on transforming a in a suitable way. To this end, we give the following lemmas.

Lemma 2.1 *Assume $\mu_0 \in M$. Then, there exists $\delta > 0$ such that, for $|\mu-\mu_0| < \delta$, we have*

$$a(\theta,\mu) = b(\theta,\mu)\prod_{j=1}^{n}[\nu_j(\mu) + \cos(\theta - \phi_j(\mu))],$$

where $2n$ is the total number of zeros of $a(\cdot,\mu_0)$ and, if $\mu \in (\mu_0-\delta,\mu_0+\delta)$, then

1. *the functions ν_j and ϕ_j, $j = 1,\dots,n$, are C^∞,*
2. *$|\nu_j(\mu)| < 1$,*
3. *b is C^∞ w.r.t. μ and $\theta \in \mathbb{T}^1$,*
4. *$b(\cdot,\mu)$ has no zeros.*

Proof: Note that, for any couple of values θ_1 and θ_2 in $\mathbb{T}^1$, $\theta_1 \neq \theta_2$, there exist values ν_0 and ϕ_0 such that the function $\nu_0 + \cos(\theta - \phi_0)$ vanishes on $\theta_{1,2}$ (these values are, in fact, $\nu_0 = \cos(\frac{1}{2}(\theta_1 - \theta_2))$ and $\phi_0 = \frac{1}{2}(\theta_1 + \theta_2) - \pi$). This also shows that if the values $\theta_{1,2}$ depend smoothly on a parameter, ν_0 and ϕ_0 also depend on the parameter with the same kind of smoothness.

Now, let us select $\delta > 0$ such that $(\mu_0 - \delta, \mu_0 + \delta) \subset M$. Then, for each μ in this interval, the number of zeros of a must be constant and equal to an even number, $2n$. Moreover, these zeros are C^∞ functions of μ. Let us group these $2n$ zeros of $a(\cdot,\mu)$ in n couples (the concrete selection is irrelevant), where each couple depends on μ in a C^∞ way. For each couple we can obtain the values $\nu_j(\mu)$ and $\phi_j(\mu)$ as above. This implies that the function

$$d(\theta,\mu) = \prod_{j=1}^{n}[\nu_j(\mu) + \cos(\theta - \phi_j(\mu))],$$

has the same zeros as a and is also C^∞ w.r.t. μ. If we define the function b as

$$b(\theta,\mu) = \frac{a(\theta,\mu)}{d(\theta,\mu)},$$

it is clear that all the statements of the lemma are satisfied. ∎

Lemma 2.2 *Assume $\mu_0 \notin M$. Under the hypotheses of Theorem 2.1, there exists $\delta > 0$ such that, for $|\mu - \mu_0| < \delta$, we have*

$$a(\theta,\mu) = b(\theta,\mu)(\nu(\mu) + \cos(\theta - \phi(\mu))),$$

where, if $\mu \in (\mu_0 - \delta, \mu_0 + \delta)$, the following statements are satisfied:

1. *b is a C^∞ function of μ and θ with simple zeros,*
2. *the functions ν and ϕ are C^∞,*
3. *$\nu(\mu_0) = 1$, $\frac{d\nu}{d\mu}(\mu_0) \neq 0$.*

Proof: As $\mu_0 \notin M$, $a(\cdot,\mu_0)$ has a double zero at, say, $\theta = \theta_0$. Then, the Malgrange Preparation Theorem ([CH82]) implies that there exists a C^∞ function q defined on an open neighbourhood of (θ_0,μ_0) such that $q(\theta_0,\mu_0) \neq 0$, and C^∞ functions d_0 and d_1 such that $d_0(\mu_0) = d_1(\mu_0) = 0$ and

$$a(\theta,\mu) = q(\theta,\mu)\left(d_0(\mu) + d_1(\mu)(\theta - \theta_0) + (\theta - \theta_0)^2\right).$$

Note that, for each μ, the function $q(\cdot,\mu)$ can be trivially extended to all $\mathbb{T}^1$, but this extension is not periodic in θ.

If we define

$$\begin{aligned} \nu(\mu) &= \cos\left(\frac{1}{2}\sqrt{d_1(\mu)^2 - 4d_0(\mu)}\right), \\ \phi(\mu) &= -\frac{1}{2}d_1(\mu) - \pi, \end{aligned}$$

it is trivial to check that, for fixed μ, the function $\nu(\mu)+\cos(\theta-\phi(\mu))$ has the same zeros as $d_0(\mu) + d_1(\mu)(\theta - \theta_0) + (\theta - \theta_0)^2$. Moreover, it is also easy to check that the functions ν and ϕ are C^∞.

Then, defining

$$b(\theta,\mu) = \frac{a(\theta,\mu)}{\nu(\mu) + \cos(\theta - \phi(\mu))}$$

it is clear that the statements of the lemma hold. ∎

Proof of Theorem 2.1: Let μ_0 be an element of M. Lemma 2.1 implies that a can be written as

$$a(\theta,\mu) = b(\theta,\mu)\prod_{j=1}^{n}[\nu_j(\mu) + \cos(\theta - \phi_j(\mu))],$$

for μ close enough to μ_0. The Lyapunov exponent of the cocycle is given by

$$\Lambda(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \ln|b(\theta,\mu)|\,d\theta + \sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi} \ln|\nu_j(\mu) + \cos(\theta - \phi_j(\mu))|\,d\theta.$$

Note that, for any τ_2, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|\tau_1 + \cos(\theta - \tau_2)|\,d\theta = \begin{cases} -\ln 2 & \text{if} \quad |\tau_1| \le 1, \\ -\ln 2 + \operatorname{arccosh}|\tau_1| & \text{if} \quad |\tau_1| \ge 1. \end{cases} \tag{14}$$

Now, using that $\int_0^{2\pi} \ln|b(\theta,\mu)|\,d\theta$ depends smoothly on μ (because $b(\theta,\mu)$ is always different from zero), the statement 1 follows.

Assume now that $\mu_0 \notin M$. For μ close enough to μ_0, Lemma 2.2 implies that

$$\Lambda(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \ln|b(\theta,\mu)|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \ln|\nu(\mu) + \cos(\theta - \phi(\mu))|\,d\theta. \tag{15}$$

We have shown above that the term $\int_0^{2\pi} \ln|b(\theta,\mu)|\,d\theta$ depends smoothly on μ. As $|\nu(\mu)|$ crosses the value 1 when μ goes through μ_0, (14) implies that Λ is only continuous at μ_0. Moreover, $\frac{d}{d\nu}\Lambda$ goes to $+\infty$ when ν goes to 1 from above, while it goes to a finite value when it goes to 1 from below. To finish the proof, note that $\Lambda' = \frac{d\Lambda}{d\nu}\frac{d\nu}{d\mu}$ and that ν is increasing when the number of zeros of a decreases at μ_0 (and vice-versa). The asymptotic expression (13) follows easily from (14) and (15).

∎

It is remarkable that the behaviour given by (13) has also been observed numerically in a two dimensional example ([HL05a]).

Corollary 2.2 ***(Normal form near a reducibility loss)*** *Let us consider the family of skew products (12). We assume that*

1. $a(\cdot,\mu)$ *is reducible for* $\mu < \mu_0$,
2. $a(\cdot,\mu)$ *has a double zero at* θ_0 *for* $\mu = \mu_0$,
3. $\frac{d}{d\mu}a(\theta_0,\mu_0) \neq 0$.

Then, there exists a neighbourhood of μ_0 *and a* C^∞ *conjugacy,*

$$\begin{aligned} y &= c(\theta,\mu)x \\ \varphi &= \theta - \phi(\mu) \\ \nu &= \nu(\mu), \end{aligned}$$

with $\nu(\mu_0) = 1$ *and* $\phi(\mu_0) = \theta_0$, *that puts the family (12) into the form*

$$\left.\begin{aligned} \bar{y} &= h(\nu)(\nu + \cos\varphi)y, \\ \bar{\varphi} &= \varphi + \omega, \end{aligned}\right\} \tag{16}$$

where h *is a smooth function that never vanishes.*

Proof: Applying Lemma 2.2 and using the transformation $\theta = \varphi + \phi(\mu)$ the cocycle takes the form

$$\left.\begin{aligned} \bar{x} &= b(\varphi,\mu)(\nu(\mu) + \cos\varphi)x, \\ \bar{\varphi} &= \varphi + \omega, \end{aligned}\right\} \tag{17}$$

where the function $b(\cdot,\mu)$ has no zeros for μ close to μ_0. Condition 3 of Lemma 2.2 implies that we can use the Inverse Function Theorem to use ν as a parameter by means of a transformation $\nu = \nu(\mu)$ (we recall that $\nu(\mu_0) = 1$). Finally, we use Proposition 2.1 to transform (17) into (16), where

$$h(\nu) = \mathrm{sign}\,(b) \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln|b(\varphi,\nu)|\,d\varphi\right].$$

The smoothness of h follows immediately from the previous expression. The smoothness of the transformation that goes from (17) to (16) is the same as for Proposition 2.1, but keeping track of the parameter ν. ■

Remark 2.1 *These techniques allow to construct a normal form –local w.r.t. μ but global w.r.t. θ– near any change of the number of zeros of a.*

2.3 The transfer operator

Definition 2.3 *If $a \in C^r(\mathbb{T}^1, \mathbb{R})$, the transfer operator $\mathcal{L} : C^r \to C^r$ is defined as*

$$(\mathcal{L}\psi)(\theta) = a(\theta - \omega)\psi(\theta - \omega) \quad \forall\,\theta \in \mathbb{T}^1. \tag{18}$$

One of the main reasons to introduce this operator here is its relation with the operator (3). It is easy to check that 0 belongs to the spectrum of (3) if and only if 1 belongs to the spectrum of (18), where $a(\theta) = D_x f_0(u_0(\theta), \theta)$. Therefore, we can apply the IFT if and only if 1 does not belong to the spectrum of the transfer operator.

It is known that the spectrum of $\mathcal{L}$ is invariant by rotations ([Mat68, HL05d]), and that if λ is an eigenvalue of $\mathcal{L}$ then, for all $k \in \mathbb{Z}$, $\exp(\mathrm{i}\,k\omega)\lambda$ is also an eigenvalue ([Jor01, HL05d]). It is also easy to see that $\mathrm{Spec}\,(\mathcal{L})$ is invariant by changes of variables $x = c(\theta)y$ in the skew-product.

Proposition 2.4 *If there exists a nontrivial closed interval I such that $a_{|I} \equiv 0$, then 0 is an eigenvalue of $\mathcal{L}$ and* $\mathrm{Spec}\,(\mathcal{L}) = \{0\}$. *If 0 is an eigenvalue of $\mathcal{L}$, then there exists a nontrivial closed interval I such that $a_{|I} \equiv 0$ and, therefore,* $\mathrm{Spec}\,(\mathcal{L}) = \{0\}$.

Proof: Assume that a vanishes on a nontrivial interval I. If ψ is a non-zero C^r function that vanishes outside I, then $\mathcal{L}\psi = 0$ and therefore ψ is an eigenfunction of eigenvalue 0. To see that the spectrum is only this value note that $\mathcal{L}$ is a nilpotent operator and, hence, its spectral radius is 0.

Assume that the spectrum of $\mathcal{L}$ reduces to the eigenvalue 0. Let ψ be a eigenfunction of eigenvalue 0, and let I be a nontrivial closed interval contained in the support of ψ. Then, a must vanish on I. ■

The reducibility of (4) can be characterised in terms of the spectrum of $\mathcal{L}$ ([Jor01]).

Proposition 2.5 *The linear skew product (4) is reducible if and only if the spectrum of $\mathcal{L}$ does not contain 0 and coincides with the closure of the set of eigenvalues.*

Proof: Assume that (4) is reducible to (5). Then, it is easy to check that the spectrum of the transfer operator for (5) is a circle of radius $|b| > 0$, and that this circle is the closure of the set of eigenvalues. To finish with this part of the proof, note that the spectrum of $\mathcal{L}$ is invariant by changes $x = c(\theta)y$ in the cocycle.

Now assume that the spectrum of $\mathcal{L}$ does not contain 0 and coincides with the closure of the set of eigenvalues. As the spectrum cannot be empty, there exists an eigenvalue/eigenfunction couple $(b, c(\theta))$, with $b \neq 0$, such that $a(\theta - \omega)c(\theta - \omega) = bc(\theta)$. From this equation it is clear that $c(\theta) \neq 0$ for all θ and, therefore, $x = c(\theta)y$ is a change of variables that transforms (4) into the reduced form (5). ∎

Remark 2.2 *It can be shown ([HL05d]) that, if a is of class $C^{r'}$, the spectrum of $\mathcal{L}$ in the C^r topology, $r \leq r'$, does not depend on r.*

2.4 Lyapunov exponents and the spectrum of transfer operators

Now we want to relate the Lyapunov exponent of a skew product with the spectral radius of its transfer operator $\mathcal{L}$. The goal is to show that if the Lyapunov exponent is negative, the hypotheses of the Implicit Function Theorem are satisfied. These properties can also be derived from the results in [CL99], that are valid in a more general context. In our particular situation their proofs become very simple, so we have included them for completeness. We stress that, in this section, ω does not need to be Diophantine but only irrational.

We note that, if $\{\theta_n\}_n$ is a sequence in $\mathbb{T}^1$ and $a \in C^0(\mathbb{T}, \mathbb{R})$ has no zeros, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta_n - j\omega)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta.$$

For the general case we only have the following inequality.

Lemma 2.3 *Let $\{\theta_n\}_n$ be a sequence in $\mathbb{T}^1$. Then,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta_n - j\omega)| \leq \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta.$$

Proof: We define

$$S_n = \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta_n - j\omega)|, \qquad S_n^{(N)} = \frac{1}{n} \sum_{j=1}^{n} \max \{\ln |a(\theta_n - j\omega)|, -N\}$$

and

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta, \qquad \Lambda^{(N)} = \frac{1}{2\pi} \int_0^{2\pi} \max\{\ln |a(\theta)|, -N\} \, d\theta.$$

for $N \in \mathbb{N}$.

It is easy to see that the triangular scheme $\{\theta_n - j\omega\}_{1\le j\le n, n\ge 1}$ is equidistributed ([Dav75], pp. 354–357). Therefore, as $\max\{\ln|a(\theta)|, -N\}$ is a continuous function of θ, we have that

$$\lim_{n\to\infty} S_n^{(N)} = \Lambda^{(N)} \qquad \forall N \in \mathbb{N}. \tag{19}$$

Now we define

$$M = \max_{\theta\in\mathbb{T}^1}\{\ln|a(\theta)|\}, \quad F(\theta) = M - \ln|a(\theta)|, \quad F_N(\theta) = M - \max\{\ln|a(\theta)|, -N\}.$$

Note that, for each θ we have $F_N(\theta) \ge 0$ and $F_N(\theta) \nearrow F(\theta)$. Then, the Lebesgue's Monotone Convergence Theorem ensures that

$$\lim_{N\to\infty} \int_0^{2\pi} F_N(\theta)\, d\theta = \int_0^{2\pi} F(\theta)\, d\theta,$$

which implies that

$$\lim_{N\to\infty} \Lambda^{(N)} = \Lambda. \tag{20}$$

We stress that the case $\Lambda = -\infty$ is included. If $\Lambda > -\infty$, eqs. (19) and (20) imply

$$\forall \varepsilon > 0 \quad \exists N_0 > 0 \text{ s.t. } \forall N \ge N_0 \text{ we have that } \limsup_{n\to\infty} S_n \le \lim_{n\to\infty} S_n^{(N)} \le \Lambda + \varepsilon.$$

This proves the lemma for the case $\Lambda > -\infty$.

If $\Lambda = -\infty$, eqs. (19) and (20) imply

$$\forall E > 0 \quad \exists N_0 > 0 \text{ s.t. } \forall N \ge N_0 \text{ we have that } \limsup_{n\to\infty} S_n \le \lim_{n\to\infty} S_n^{(N)} \le -E.$$

This proves the lemma for $\Lambda = -\infty$. ∎

Theorem 2.2 *Let $\mathcal{L} : C^0 \to C^0$ and Λ denote, respectively, the transfer operator and the Lyapunov exponent of (4). Then,*

$$\rho(\mathcal{L}) = \exp(\Lambda).$$

Proof: As $\rho(\mathcal{L}) = \lim_{n\to\infty} \|\mathcal{L}^n\|_\infty^{\frac{1}{n}}$, we have that

$$\rho(\mathcal{L}) = \lim_{n\to\infty} \left(\max_{\theta\in\mathbb{T}^1} \prod_{j=1}^{n} |a(\theta - j\omega)| \right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\prod_{j=1}^{n} |a(\theta_n - j\omega)| \right)^{\frac{1}{n}}, \tag{21}$$

where θ_n is a value for which the maximum is attained. Now Lemma 2.3 implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln|a(\theta - j\omega)| \le \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln|a(\theta_n - j\omega)| \le \Lambda, \qquad \text{for all } \theta \in \mathbb{T}^1,$$

If $\Lambda = -\infty$, the previous "$\leq$" become "$=$" and the proof is finished. If $\Lambda > -\infty$, the Birkhoff Ergodic Theorem implies that there exists a set of values of θ, with total Lebesgue measure, such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta - j\omega)| = \Lambda.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta_n - j\omega)| = \Lambda.$$

Taking exponentials at both sides, we obtain the desired result. ∎

Corollary 2.3 *If $\Lambda = -\infty$, then $\operatorname{Spec} \mathcal{L} = \{0\}$. Moreover, if a vanishes on a non-degenerate interval, 0 is an eigenvalue of $\mathcal{L}$; otherwise 0 is an spectral value but not an eigenvalue.*

Proof: Use Proposition 2.4 and Theorem 2.2. ∎

Next result can also be derived from the more general results in [HL05c]. We have included a different proof, taking advantage of the particularities of our case.

Theorem 2.3 *Assume that the function a in (4) has zeros and that $\mathcal{L}$ acts on C^0. Then,*

$$\operatorname{Spec}(\mathcal{L}) = \{z \in \mathbb{C} \textit{ such that } |z| \leq \exp(\Lambda)\}.$$

Proof: We will assume that $\operatorname{Spec}(\mathcal{L}) \neq \{0\}$ (otherwise the result is trivial). As the spectrum is invariant by rotations, it is enough to consider resolvents $\mathcal{L} - \lambda \operatorname{Id}$ for λ real and positive.

We will proceed by contradiction: Let us take a fixed value $0 < \lambda < \exp(\Lambda)$ (the case $\lambda = \exp(\Lambda)$ follows immediately from Theorem 2.2), and assume that $\lambda \notin \operatorname{Spec}(\mathcal{L})$. Then, the Open Mapping Theorem implies that the operator $\mathcal{L} - \lambda \operatorname{Id}$, acting on the space of continuous functions endowed with the sup norm, has a bounded inverse.

Let $A \subset \mathbb{T}^1$ be the set of values of θ for which the Birkhoff Ergodic Theorem applies, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \ln |a(\theta + j\omega)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta = \Lambda, \qquad \text{for all } \theta \in A.$$

In particular, the zeroes of a are excluded from A.

Let $b \in C^0(\mathbb{T}^1, \mathbb{R})$ such that $\|b\| \leq 1$, and let us define $\psi_b = (\mathcal{L} - \lambda \operatorname{Id})^{-1} b$. Note that $\|\psi_b\| \leq \|(\mathcal{L} - \lambda \operatorname{Id})^{-1}\|$. This implies that ψ_b satisfies the equation

$$\psi_b(\theta + \omega) = \frac{1}{\lambda}(a(\theta)\psi_b(\theta) - b(\theta + \omega)), \qquad \text{for all } \theta \in \mathbb{T}^1.$$

Therefore,

$$\begin{aligned}\psi_b(\theta+n\omega) &= \frac{1}{\lambda^n}a(\theta+(n-1)\omega)\cdots a(\theta)\psi_b(\theta)\\ &\quad -\frac{1}{\lambda^n}a(\theta+(n-1)\omega)\cdots a(\theta+\omega)b(\theta+\omega)\\ &\quad \vdots\\ &\quad -\frac{1}{\lambda^2}a(\theta+(n-1)\omega)b(\theta+(n-1)\omega)-\frac{1}{\lambda}b(\theta+n\omega).\end{aligned}$$

For any $\theta\in A$, we can rewrite the previous expression as

$$\begin{aligned}\psi_b(\theta) &= \frac{b(\theta+\omega)}{a(\theta)}+\lambda\frac{b(\theta+2\omega)}{a(\theta)a(\theta+\omega)}+\cdots+\lambda^{n-1}\frac{b(\theta+n\omega)}{a(\theta)\cdots a(\theta+(n-1)\omega)}\\ &\quad +\lambda^n\frac{\psi_b(\theta+n\omega)}{a(\theta)\cdots a(\theta+(n-1)\omega)}.\end{aligned} \tag{22}$$

Note that, for all $\theta\in A$,

$$\lim_{n\to\infty}\frac{\lambda^n}{a(\theta)a(\theta+\omega)\cdots a(\theta+(n-1)\omega)}=0. \tag{23}$$

Let θ_* be a zero of a. Let $\{\theta_k\}_k$ be a sequence of elements of A that converges to θ_*, and that $|a(\theta_k)|<\frac{1}{k}$. We consider (22) for $\theta=\theta_k$ and, for each k, (23) implies that we can select $n=n_k$ such that

$$\frac{|a(\theta_k)\cdots a(\theta_k+(n_k-1)\omega)|}{\lambda^{n_k}}>\|(\mathcal{L}-\lambda\,\mathrm{Id})^{-1}\|.$$

Besides, for each k, we take $b_k\in C^0(\mathbb{T}^1,\mathbb{R})$ such that $\|b\|=1$ and that the following conditions are met:

$$\begin{aligned}b_k(\theta_k+\omega) &= \mathrm{sign}\,(a(\theta_k)),\\ b_k(\theta_k+2\omega) &= \mathrm{sign}\,(a(\theta_k)a(\theta_k+\omega)),\\ &\vdots\\ b_k(\theta_k+n_k\omega) &= \mathrm{sign}\,(a(\theta_k)a(\theta_k+\omega)\cdots a(\theta_k+(n_k-1)\omega)),\end{aligned}$$

where $\mathrm{sign}\,(x)$ is 1 if $x>0$ and -1 if $x<0$. Obviously, such functions b_k exist although the sequence $\{b_k\}_k$ does not need to have limit in C^0. Now observe that

$$\left|\frac{\lambda^{n_k}\psi_{b_k}(\theta+n_k\omega)}{a(\theta_k)\cdots a(\theta_k+(n_k-1)\omega)}\right|\le 1.$$

Using (22) for $\theta=\theta_k$, we have that

$$\frac{1}{|a(\theta_k)|}-1\le|\psi_{b_k}(\theta_k)|\le\|(\mathcal{L}-\lambda\,\mathrm{Id})^{-1}\|.$$

If k is large enough, the previous formula implies that $\|(\mathcal{L}-\lambda\,\mathrm{Id})^{-1}\|$ is unbounded, which contradicts a previous statement. ∎

Corollary 2.4 *Assume $a \in C^r$ and consider $\mathcal{L} : C^r \to C^r$, for $0 \leq r < \infty$. Then, Theorem 2.3 still holds.*

Proof: Use that the spectrum does not depend on the considered value of r (see [HL05d]). ■

Remark 2.3 *In Theorem 2.3 we cannot substitute the hypothesis on the existence of zeros of the function a by the hypothesis of non reducibility. Indeed, as we proved in Proposition 2.2 there exist functions a without zeros for which the associated skew product is not reducible. It is easy to prove that in this case*

$$\mathrm{Spec}\,(\mathcal{L}) = \{z \in \mathbb{C} \text{ such that } |z| = \exp(\Lambda)\}$$

and there are not eigenvalues in the spectrum.

In summary, we have shown that if a C^r invariant curve is attracting, it can be locally continued with respect to the parameter μ. We have also shown that the IFT cannot be applied to a repelling and non reducible invariant curve.

3 Fractalization in affine systems

In this section we will focus on dynamical systems of the form

$$\left.\begin{array}{rcl} \bar{x} &=& \alpha\, a(\theta)x + b(\theta), \\ \bar{\theta} &=& \theta + \omega, \end{array}\right\} \tag{24}$$

where a and b are C^r functions and α is a real positive parameter. We are interested in knowing the range of values of α for which there exists an attracting invariant curve, and in the behaviour of this curve when α approaches the boundary of this range.

It is clear that the linearised normal behaviour around an invariant curve of (24) is described by

$$\left.\begin{array}{rcl} \bar{x} &=& \alpha\, a(\theta)x, \\ \bar{\theta} &=& \theta + \omega. \end{array}\right\} \tag{25}$$

Although the main results of this section are for the non reducible case, some of them are valid when (25) is reducible. Therefore, we will write explicitly the concrete assumptions for each result. The Lyapunov exponent of (25) is given by

$$\Lambda = \ln \alpha + \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)|\, d\theta.$$

If the integral above exists (and it is finite), then the Lyapunov exponent is negative for sufficiently small values of α, namely,

$$\alpha < \alpha_0 = \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)|\, d\theta\right).$$

In particular this implies that, for $\alpha < \alpha_0$, any invariant curve of (24) is globally attracting and, therefore, the invariant curve must be unique.

The goal of this section is to discuss the behaviour of this curve w.r.t. α. The existence of the curve is shown in Section 3.1. If the curve is not reducible (and under some extra hypotheses) we will prove in Section 3.2 that, when $\alpha \nearrow \alpha_0$ (i.e., the Lyapunov exponent goes to zero from below), the curve undergoes a fractalization process. On the other hand, for $\alpha > \alpha_0$, Section 3.3 shows that there is no continuous repelling curve. These results will be used in Section 4.1 to show an example of attracting curve that looks like a strange set.

3.1 On the existence of attracting curve

For the moment being, assume that there exists an invariant curve of (24), that we will denote as $x_\alpha(\theta)$. Let us focus on the formal expression

$$\begin{aligned} x_\alpha(\theta) &= b(\theta-\omega) + \alpha\, a(\theta-\omega)b(\theta-2\omega) + \alpha^2\, a(\theta-\omega)\, a(\theta-2\omega)b(\theta-3\omega) \\ &\quad +\alpha^3\, a(\theta-\omega)\, a(\theta-2\omega)\, a(\theta-3\omega)b(\theta-4\omega) + \cdots \\ &= b(\theta-\omega) + \sum_{n=1}^{\infty} \alpha^n \left(\prod_{j=1}^{n} a(\theta-j\omega) \right) b(\theta-(n+1)\omega). \end{aligned} \tag{26}$$

A simple calculation shows that this formal expression satisfies equation (24) so it is clear that if it converges, it defines an invariant curve. The convergence can be discussed by using the root criterion: as b is bounded, a necessary and sufficient condition to have point-wise convergence is

$$\limsup_{n\to\infty} \left(\max_{\theta\in\mathbb{T}^1} \prod_{j=1}^{n} |a(\theta-j\omega)| \right)^{\frac{1}{n}} < \frac{1}{\alpha}. \tag{27}$$

A crude estimate of the convergence radius comes from bounding $|a(\theta)|$ by its sup norm:

$$\limsup_{n\to\infty} \left(\max_{\theta\in\mathbb{T}^1} \prod_{j=1}^{n} |a(\theta-j\omega)| \right)^{\frac{1}{n}} \le \limsup_{n\to\infty} \left(\|a\|_\infty^n\right)^{\frac{1}{n}} = \|a\|_\infty.$$

Therefore, for $\alpha < \|a\|_\infty$, the series converges uniformly to a continuous invariant curve of (24).

In this case, the transfer operator $\mathcal{L}_\alpha : C^r \to C^r$ (see (18)) is given by

$$(\mathcal{L}_\alpha \psi)(\theta) = \alpha a(\theta-\omega)\psi(\theta-\omega). \tag{28}$$

Note that if (26) defines a function $x(\theta)$, then this function must satisfy $(\mathcal{L}_\alpha - \mathrm{Id})x(\theta) = -b(\theta-\omega)$. In fact, the series in (26) is the result of applying to $-b(\theta-\omega)$ the Neumann series that (formally) defines the inverse of the operator $\mathcal{L}_\alpha - \mathrm{Id}$.

As before, if we denote by $\rho(\mathcal{L}_\alpha)$ the spectral radius of $\mathcal{L}_\alpha$, it is clear that

$$\rho(\mathcal{L}_\alpha) = \alpha\rho(\mathcal{L}_1). \tag{29}$$

Moreover, Theorem 2.2 implies that

$$\rho(\mathcal{L}_1) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|a(\theta)|\,d\theta\right) = \frac{1}{\alpha_0}. \tag{30}$$

Proposition 3.1 *If a and b are of class C^r and $\alpha < \alpha_0$, then the series (26) converges to the unique attracting invariant curve of class C^r of (24).*

Proof: As $\alpha < \alpha_0$, $1 \notin \operatorname{Spec}(\mathcal{L}_\alpha)$ and, therefore, there exists a unique function x of class C^r such that $(\mathcal{L}_\alpha - \mathrm{Id})x(\theta) = -b(\theta - \omega)$. To show the convergence, note that (27) holds due to (21). ■

These results are true regardless of the reducibility of (25). If we assume that (25) is reducible (which implies that a has no zeroes) and that $\alpha > \alpha_0$, we can apply the previous results to the inverse of the map (24) to show that, in this case, there exists a unique repelling invariant curve of class C^r. As we will see in Section 3.3, repelling curves does not need to exist if we remove the reducibility condition. This is an important difference between reducible and non reducible cases.

3.2 The fractalization mechanism

Now we need a rigorous definition for the word "fractalization". We note that the computation of the Hausdorff dimension cannot detect that a smooth curve is "becoming fractal" (because it takes the value 1 as long as the object is a curve). A better option is to consider that a curve is undergoing a fractalization process for $\alpha \to \alpha_0$ when the curve keeps bounded while the lim sup of the derivatives is unbounded. In this section, as we are dealing with an affine system and the sup norm of a curve does not need to be bounded, we will say that a curve is fractalizing when its C^1 norm –taken on any closed nontrivial interval for θ– goes to infinity much faster than its C^0 norm, that is, when

$$\limsup_{\alpha\to\alpha_0} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty, \tag{31}$$

where $\|\cdot\|_{I,\infty}$ denotes the sup norm on a nontrivial closed interval I. As we will see later, this definition is very suitable to explain the results of the numerical simulations as well as to derive rigorous proofs.

Let us start by introducing some notations. For $\alpha < \alpha_0$, and for a given continuous function $b(\theta)$, let us denote by $x_\alpha(\theta)$ the solution of (24). We recall that a residual set is defined as the countable intersection of dense open sets. Next result does not depend on reducibility.

Proposition 3.2 *Assume that $a \in C^r(\mathbb{T},\mathbb{R})$ for a given $r \geq 0$. Then, there exists a residual set $D_r \subset C^r(\mathbb{T},\mathbb{R})$ such that, if $b \in D_r$, we have*

$$\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_{C^r} = +\infty. \tag{32}$$

Proof: Note that the map $\alpha \to (\mathcal{L}_\alpha - \mathrm{Id})^{-1}$ is continuous for $0 < \alpha < \alpha_0$ and, hence, the map $\alpha \to \|x_\alpha\|_{C^r}$ is also continuous. This implies that (32) is equivalent to

$$\sup_{\alpha \in [\alpha_1, \alpha_0)} \|x_\alpha\|_{C^r} = +\infty, \quad \text{for any} \quad \alpha_1 > 0. \tag{33}$$

As $(\mathcal{L}_\alpha - \mathrm{Id})^{-1} = \frac{1}{\alpha}(\mathcal{L}_1 - \frac{1}{\alpha}\mathrm{Id})^{-1}$, we can apply the Representation Theorem for the Resolvent ([Kre78]) to obtain

$$\|(\mathcal{L}_\alpha - \mathrm{Id})^{-1}\| \geq \frac{1}{\alpha} \left| \frac{1}{\alpha_0} - \frac{1}{\alpha} \right|^{-1}, \qquad \text{for all } \alpha \in [\alpha_1, \alpha_0).$$

This implies that $\|(\mathcal{L}_\alpha - \mathrm{Id})^{-1}\|$ is not bounded for $\alpha \in [\alpha_1, \alpha_0)$. Therefore, using the Banach-Steinhaus Theorem (also called principle of uniform boundedness; see [Rud74]) it follows that (33) holds for b belonging to a suitable residual set. ∎

If $x \in C^r(\mathbb{T}, \mathbb{R})$ and $I \subset \mathbb{T}$ is a closed interval, we define $\|x\|_{I,C^r}$ as the usual C^r-norm of x restricted to the subset I.

Corollary 3.1 *For any nontrivial closed interval $I \subset \mathbb{T}$, if $b \in D_r$ then*

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,C^r} = +\infty \tag{34}$$

Proof: As before, the map $\alpha \to \|x_\alpha\|_{I,C^r}$ is continuous for $0 < \alpha < \alpha_0$ and, hence, (34) is equivalent to

$$\sup_{\alpha \in [\alpha_1, \alpha_0)} \|x_\alpha\|_{I,C^r} = +\infty, \quad \text{for any} \quad \alpha_1 > 0. \tag{35}$$

Assume that there exists $b \in D_r$ and a nontrivial closed interval $I \subset \mathbb{T}$ such that the sup in (35) is finite. As ω is irrational, any value $\theta \in \mathbb{T}$ can be obtained from a value in I by adding a (bounded) multiple of ω. This implies that there exists constants K_1 and K_2, depending on ω, a, b and I such that

$$\|x_\alpha\|_{C^r} \leq K_1 \|x_\alpha\|_{I,C^r} + K_2. \tag{36}$$

This contradicts the assumption $b \in D_r$. ∎

Next result is the key result of this section.

Theorem 3.1 *Assume that $a, b \in C^1(\mathbb{T}, \mathbb{R})$ and that (25) is not reducible. Then,*

a) If

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty,$$

and $b \in D_1$ (D_1 is the residual set introduced in Proposition 3.2), we have

$$\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty,$$

for any nontrivial closed interval $I \subset \mathbb{T}$.

b) If

$$\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_\infty = +\infty,$$

then, for any nontrivial closed interval $I \subset \mathbb{T}$, we have

$$\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \qquad \text{and} \qquad \limsup_{\alpha\to\alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

Proof: Assume that $\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_\infty < +\infty$. Therefore, for any closed interval $I \subset \mathbb{T}$, we have $\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_{I,\infty} < +\infty$. Then, using $k = 1$ in Corollary 3.1 we obtain $\limsup_{\alpha\to\alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty$. On the other hand, if $\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_\infty = +\infty$, we can use (36) for $k = 0$ to show that $\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty$.

Now, for $\alpha < \alpha_0$, we apply the change of variables $x = \|x_\alpha\|_\infty y$ to (24) to obtain

$$\left.\begin{array}{rcl} \bar{y} &=& \alpha\, a(\theta)\, y + \dfrac{b(\theta)}{\|x_\alpha\|_\infty}, \\ \bar{\theta} &=& \theta + \omega, \end{array}\right\} \tag{37}$$

Note that (37) has the invariant curve $y_\alpha(\theta) = \frac{x_\alpha(\theta)}{\|x_\alpha\|_\infty}$. Now we proceed by contradiction: if we assume that there exists $I \subset \mathbb{T}^1$ such that $\limsup_{\alpha\to\alpha_0} \|y'_\alpha\|_{I,\infty} < +\infty$ then, as there exist positive constants $K_{1,2}$ such that $\|y'_\alpha\|_\infty \leq K_1\|y'_\alpha\|_{I,\infty} + K_2$, we have that $\limsup_{\alpha\to\alpha_0} \|y'_\alpha\|_\infty < +\infty$. By the Ascoli theorem, there exists a sequence $\{y_{\alpha_n}\}_{n>0}$ (with $\alpha_n \to \alpha_0$) that converges uniformly to a continuous function y_{α_0} with $\|y_{\alpha_0}\|_\infty = 1$. Therefore, $y_{\alpha_0}(\theta+\omega) = \alpha_0 a(\theta) y_{\alpha_0}(\theta)$, which means that α_0 is an eigenvalue of the transfer operator (28). This contradicts the fact that the invariant curve is not reducible. ∎

Note that, if $b \in D_1$, this proposition implies (no matter if we are in the first or second case) that (31) holds and, therefore, that the curve is undergoing a fractalization process. A numerical example will be shown in Section 4.1.

If we assume in Theorem 3.1 that (25) is reducible, the situation is different. For instance, if we add the assumptions that ω is Diophantine and that a, b are C^r for r large enough, neither a) nor b) are true. In this case, it is not difficult to see that

a) If $\displaystyle\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_\infty < +\infty$ then $\displaystyle\limsup_{\alpha\to\alpha_0^-} \|x'_\alpha\|_\infty < +\infty$.

b) If $\displaystyle\limsup_{\alpha\to\alpha_0^-} \|x_\alpha\|_\infty = +\infty$ then $\displaystyle\limsup_{\alpha\to\alpha_0^-} \frac{\|x'_\alpha\|_\infty}{\|x_\alpha\|_\infty} < +\infty$.

3.3 Non-existence of repelling continuous curves

In this section we assume that $\alpha > \alpha_0$ which implies that the origin of (25) is a repellor. This also implies that $a(\theta)$ has to be different from 0 a.e. (with respect to the Lebesgue measure). As before, we are assuming that (25) has a zero (so it is not reducible) and we are interested in the existence of a repelling invariant curve for (24). We stress that the results in this section are false if (25) is reducible.

Proposition 3.3 *Assume, for all $\theta \in \mathbb{T}^1$, that $a(\theta) \geq 0$ and that there exists a value θ_0 such that $a(\theta_0) = 0$. Then the operator*

$$x(\theta) \mapsto x(\theta + \omega) - \alpha a(\theta) x(\theta),$$

defined on $C^0(\mathbb{T}^1, \mathbb{R})$, is not surjective. In particular, there is no $x \in C^0(\mathbb{T}^1, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + 1$.

Proof: Let us select $b(\theta) \equiv 1$ and assume that there exist a continuous function $x(\theta)$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + 1$. Note that $a(\theta_0) = 0$ implies $x(\theta_0 + \omega) = 1$. On the other hand, let θ_1 be a value for which the Ergodic Theorem applies to $\ln(a(\theta))$. Then, the fact that the Lyapunov exponent is positive implies that, if $x(\theta_1) > 0$, the sequence $\{x(\theta_1 + n\omega)\}_n$ is not bounded which contradicts the continuity of x. Therefore, we must assume that $x(\theta_1) \leq 0$. Note that the Ergodic Theorem is valid for a dense set of values θ_1 which implies that $x(\theta) \leq 0$ for all θ. This contradicts the existence of θ_0 such that $x(\theta_0 + \omega) = 1$. ■

Proposition 3.4 *Assume, in the hypothesis of Proposition 3.3, that $a(\theta)$ is not always positive. Moreover, let us assume that $a \in C^r(\mathbb{T}^1, \mathbb{R})$ for a given $r \geq 0$. Then, there exists $b \in C^r(\mathbb{T}^1, \mathbb{R})$ for which there is no $x \in C^r(\mathbb{T}^1, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$.*

Proof: The equation $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$ can be rewritten as

$$(\mathcal{L}_\alpha - \mathrm{Id}) x(\theta) = -b(\theta - \omega), \tag{38}$$

where we recall that $\mathcal{L}_\alpha x(\theta) \equiv \alpha a(\theta - \omega) x(\theta - \omega)$ is the so-called transfer operator. Using (29) and (30) we have that

$$\rho(\mathcal{L}_\alpha) = \frac{\alpha}{\alpha_0} > 1,$$

and then, Theorem 2.3 implies that $1 \in \mathrm{Spec}\,(\mathcal{L}_\alpha)$. Therefore, there exist functions b such that (38) cannot be solved. ■

These results show that, when a has zeros, the repelling situation is very different from the attracting one: while attracting curves are "robust" and can be locally continued w.r.t. parameters, repelling curves are "isolated" and do not survive generic perturbations.

3.4 A particular situation

In this section we focus on the fractalization phenomena for the affine system (24), but assuming that a is a positive function with at least a zero, so that the skew product is still not reducible. As before, if (24) has an invariant curve for a given value of α, it will be denoted by x_α.

Proposition 3.5 *Assume, in (24), that $a, b \in C^1(\mathbb{T}, \mathbb{R})$, $a(\theta) \geq 0$ for all $\theta \in \mathbb{T}^1$ and there exists a value θ_0 such that $a(\theta_0) = 0$. We also assume that b never vanishes. Then,* *a) If $a, b \in C^r(\mathbb{T}, \mathbb{R})$, $r \geq 1$, then $x_\alpha \in C^r(\mathbb{T}, \mathbb{R})$ for $0 < \alpha < \alpha_0$.*

b) For any nontrivial closed interval $I \subset \mathbb{T}$, we have

$$\lim_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \qquad \text{and} \qquad \lim_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

c) For $\alpha > \alpha_0$, there is no $x \in C^0(\mathbb{T}, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$.

Proof: Item a) follows from Proposition 3.1. To see item b), we denote by θ_n a value of θ such that

$$\max_{\theta \in \mathbb{T}^1} \prod_{j=1}^{n} a(\theta - j\omega) = \prod_{j=1}^{n} a(\theta_n - j\omega).$$

Then,

$$\|(\mathcal{L}_{\alpha_0} - \mathrm{Id})^{-1}\| \leq \sum_{n=0}^{\infty} \alpha_0^n \prod_{j=1}^{n} a(\theta_n - j\omega),$$

and using that $1 \in \operatorname{Spec} \mathcal{L}_{\alpha_0}$, we have

$$\sum_{n=0}^{\infty} \alpha_0^n \prod_{j=1}^{n} a(\theta_n - j\omega) = +\infty.$$

On the other hand, for $0 < \alpha < \alpha_0$ we have that

$$x_\alpha(\theta) = b(\theta - \omega) + \sum_{n=1}^{\infty} \alpha^n \left(\prod_{j=1}^{n} a(\theta - j\omega) \right) b(\theta - (n+1)\omega),$$

where we note that, as a is positive and b has constant sign, all the terms in these sums have the same sign. Therefore, using that $|b(\theta)| \geq \beta > 0$, we have

$$\|x_\alpha\|_\infty \geq \beta + \sum_{n=1}^{\infty} \alpha^n \left(\prod_{j=1}^{n} a(\theta_n - j\omega) \right) \beta,$$

which goes to infinity when α goes to α_0. Let us select a nontrivial interval $I \subset \mathbb{T}^1$. Then, Theorem 3.1 implies that

$$\limsup_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

Following the proof of Theorem 3.1, it is not difficult to check that, in this case (we have a lim for the norms of x_α), this lim sup can be replaced by a lim.

Finally, to prove item c), we apply $x = b(\theta - \omega) y$ to (24) to obtain

$$\left.\begin{array}{rcl} \bar{y} &=& \alpha \dfrac{b(\theta - \omega)}{b(\theta)} a(\theta) y + 1, \\ \bar{\theta} &=& \theta + \omega. \end{array}\right\}$$

Then, from Proposition 3.3 we know that this system does not have any continuous invariant curve. This finishes the proof. ■

We have seen in Theorem 3.1 that fractalization occurs for b in a suitable residual set. In the last proposition, we have shown that for any $a \geq 0$ (but with at least a zero), fractalization appears for *any* nonvanishing b. We note that the set of nonvanishing function b is open and can be much larger than a residual set. Therefore, we believe that fractalization is a common phenomenon in several contexts. We have included some examples in the next section.

4 Applications

Here we have included some numerical examples. One of the main issues of this section is to show that fractalization is a process that needs a careful numerical treatment. As we will see, there are examples of smooth curves that look like fractal sets.

4.1 Fractalization in affine systems

We will focus in two examples. The first one is

$$\left.\begin{array}{rcl} \bar{x} &=& \alpha\,(1+\cos\theta)x+1, \\ \bar{\theta} &=& \theta+\omega, \end{array}\right\}$$

where ω is the golden mean. This example satisfies the hypotheses of Proposition 3.5 and from (14) we immediately obtain that the Lyapunov exponent of the linear skew product is $\Lambda = \ln\alpha - \ln 2$ and, therefore, the critical value α_0 is 2. Then, there exists a unique invariant attracting curve for $0 < \alpha < 2$, that undergoes a fractalization process when $\alpha \to 2^-$. Figure 1 shows this curve, for α equal to 1.99 and 1.999, where it is seen that the derivative goes to infinity faster than the curve. We stress that, although it looks very twisted, the curve is of class C^∞ as long as $\alpha < 2$. We have also proved that, for $\alpha > 2$, there is no invariant (and repelling) curve of class C^0 in this system.

In the second example we also consider (24) but now with $a(\theta) \equiv \cos\theta$ and $b(\theta) \equiv 1$. As $a(\theta)$ has no constant sign, this case is not covered by Proposition 3.5. As in the previous example, the Lyapunov exponent of the corresponding skew-product (25) is $\Lambda = \ln\alpha - \ln 2$. Hence, for $\alpha < 2$ (and only for this case), the Lyapunov exponent is negative.

First, let us focus on the case $\alpha < 2$. In Section 3.1, we have seen that there is a unique attracting invariant curve, that can be written as

$$\begin{array}{rcl} x_\alpha(\theta) &=& 1+\alpha\cos(\theta-\omega)+\alpha^2\cos(\theta-\omega)\cos(\theta-2\omega) \\ && +\alpha^3\cos(\theta-\omega)\cos(\theta-2\omega)\cos(\theta-3\omega)+\cdots \\ &=& \displaystyle\sum_{n=0}^{\infty}\alpha^n\prod_{j=1}^{n}\cos(\theta-j\omega). \end{array}$$

This curve is plotted in Figure 2, for several values of the parameter α. Looking at the first 4 plots, it seems that the sup norm of the curve goes to infinity. Moreover, as the

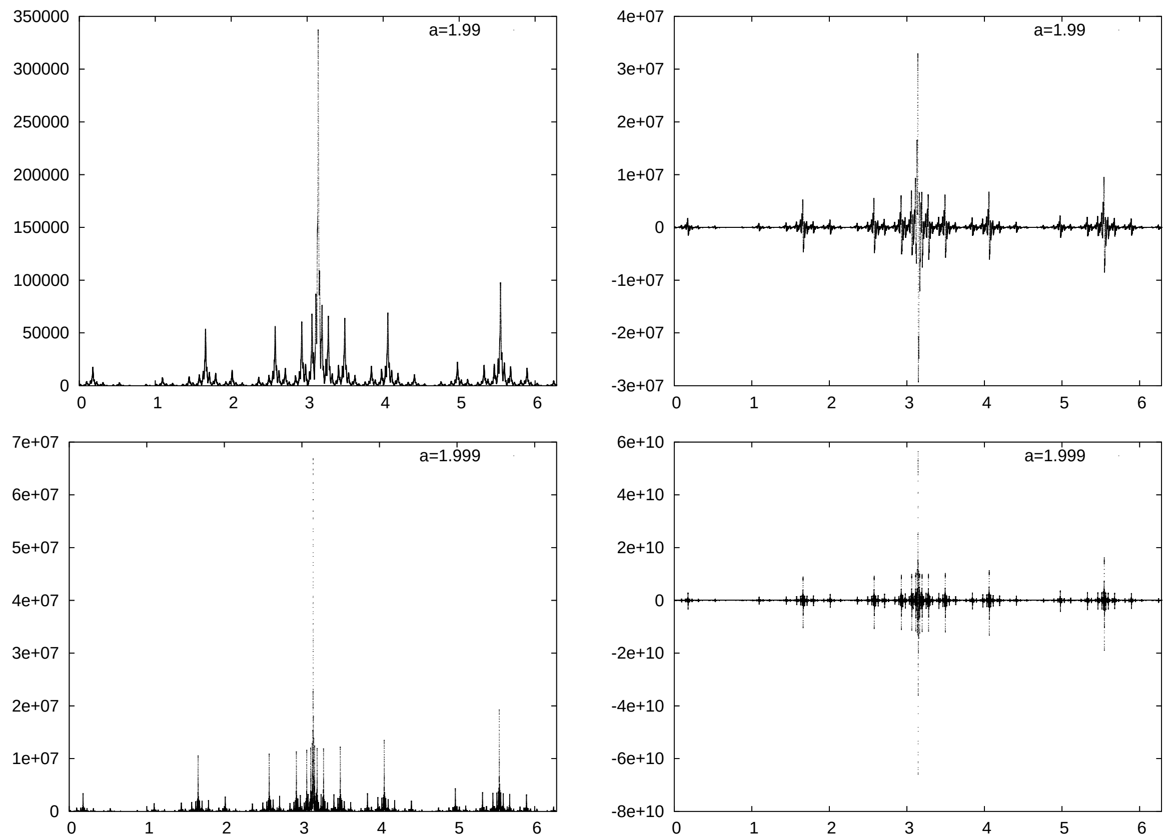

Figure 1: Attracting invariant curve of (25) for $a(\theta) \equiv 1 + \cos\theta$, $b(\theta) \equiv 1$ and for α equal to 1.99 (first row) and 1.999 (second row). The first column displays the attracting curve and the second column shows its derivative.

vertical "size" of the plots is always the same, we can look at them as if we were plotting the "normalised" curve $\frac{x_\alpha(\theta)}{\|x_\alpha(\cdot)\|_\infty}$. The first plot of the last row is a magnification for the case $\alpha = 1.999$, and shows the "wild" behaviour of this C^∞ curve. In the last plot we show the derivative of this curve. Note that the derivative seems to go to infinity much faster than the function.

In principle, we cannot apply item a) of Theorem 3.1 to this example because we do not know whether the constant function 1 belongs to the residual set D_1. If this were the case, then this theorem would imply that the curve is undergoing a fractalization procedure. However, if we accept that $\limsup_{\alpha\to 2^-} \|x_\alpha\|_\infty$ is going to infinity (which is what the numerical simulations seem to indicate), then item b) of Theorem 3.1 also implies that the curve must be fractalizing. We stress that, when α approaches the critical value 2, these smooth curves look as strange non-chaotic attractors, and it can be extremely difficult to detect them as curves by numerical simulation.

If $\alpha > 2$, Proposition 3.4 implies that there exists continuous functions b for which (24) does not have a continuous (and repelling) invariant curve. Again, we do not know if 1 is one of these functions so we will simply try to iterate the inverse map of this example,

$$\left.\begin{array}{rcl} x & = & \dfrac{\bar{x}-1}{\alpha\cos(\bar{\theta}-\omega)}, \\[2ex] \theta & = & \bar{\theta}-\omega, \end{array}\right\} \qquad (39)$$

If this has an attracting (and smooth) curve, then (24) will have a repelling one in the same smoothness class. We select initial conditions (x_0, θ_0) such that $\cos(\bar{\theta}-\omega)$ is never 0, and we iterate the map and, after some transient, we plot the attracting set (see Figure 3). From these plots, it seems that the attracting set is not a continuous curve. Besides of the apparent discontinuities, the attracting set also seems unbounded (that is the reason to only plot a magnification for the x coordinate). For instance, the diameter of the computed attracting set, after 1.1×10^6 iterates, is of the order of 5×10^5 for the case $\alpha = 2.1$, and of the order of 4×10^5 for the case $\alpha = 3.0$.

4.2 An example by G. Keller

It is interesting to apply these results to a well-known example by Keller ([Kel96]). In this work, the author rigorously proves the existence of Strange Non-chaotic Attractors (SNAs) for systems of the form

$$\left.\begin{array}{rcl} \bar{x} & = & f(x)g(\theta), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\} \qquad (40)$$

where $f : [0,\infty) \to [0,\infty)$ is increasing, strictly concave, bounded in C^1 norm and satisfies $f(0) = 0$, while $g : \mathbb{T}^1 \to [0,\infty)$ is only continuous. We define

$$\sigma = f'(0)\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln|g(\theta)|\,d\theta\right), \qquad (41)$$

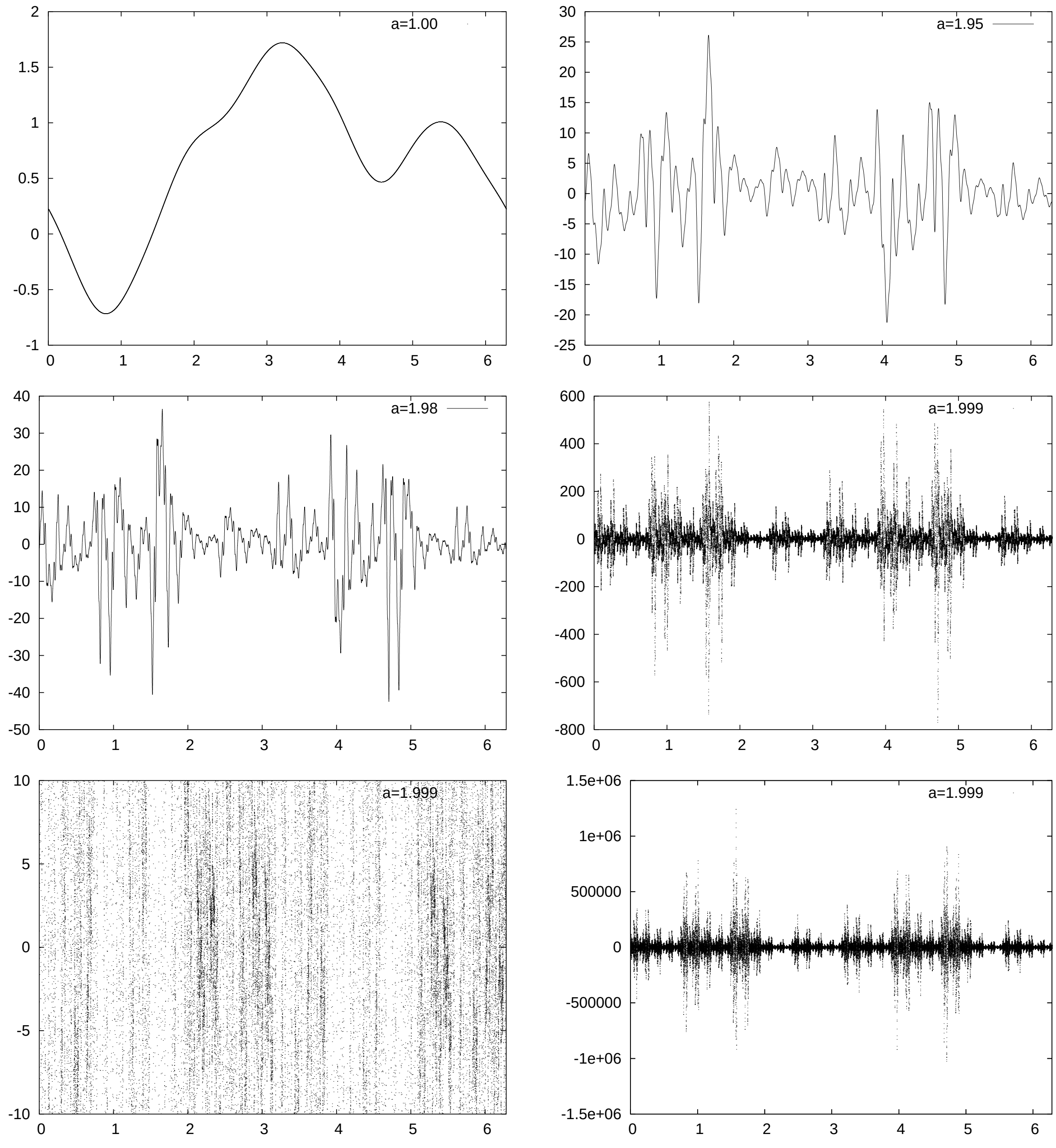

Figure 2: Attracting invariant curve of (25) for $a(\theta) \equiv \cos\theta$, $b(\theta) \equiv 1$ and for α equal to 1, 1.95, 1.98, 1.999. The first plot in the last row is a magnification of the case $\alpha = 1.999$, and the last plot shows the derivative (w.r.t. θ) of this invariant curve.

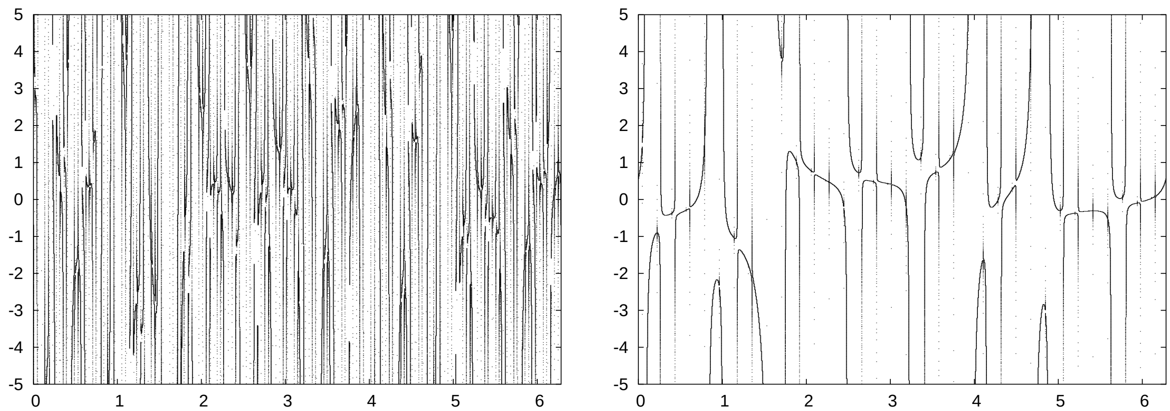

Figure 3: Magnification of the attracting set of (39). Left: $\alpha = 2.1$. Right: $\alpha = 3.0$.

where we set $\sigma = 0$ if the above integral is $-\infty$. In ([Kel96]) it is proved that (40) has a SNA provided that there exists at least a value $\hat{\theta}$ such that $g(\hat{\theta}) = 0$ and $\sigma > 1$.

It is very interesting to look at this result from a bifurcation point of view. To this end, we will introduce a parameter α by replacing the function f in (40) by αf. Therefore, the new value of σ is $\sigma_\alpha = \alpha\sigma$, where σ has been defined in (41). For simplicity, we will consider σ_α as the parameter of the system. Following the results in ([Kel96]) it is clear that, for $\sigma_\alpha < 1$, $x = 0$ is an attracting invariant curve (its vertical Lyapunov exponent is $\ln \sigma_\alpha < 0$). It is remarkable that, as the function g has at least one zero, $x = 0$ is a non reducible curve. When σ_α increases and crosses the critical value 1, the Lyapunov exponent of the origin changes from negative to positive, so that $x = 0$ becomes a repelling curve. We recall that, for $\sigma_\alpha = 1$, the IFT cannot be applied to guarantee the local continuation of the curve. In fact, what happens is that the non reducible curve $x = 0$ undergoes a bifurcation and a SNA branches off. It is also clear that, due to the specific properties of this map, the set $x = 0$ cannot become fractal or disappear when its Lyapunov exponent becomes positive. Therefore, this example fits perfectly with our results.

An interesting modification of this situation is given by the following example:

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha \cos\theta \tanh x + \tau, \\ \bar{\theta} & = & \theta + \omega. \end{array}\right\} \qquad (42)$$

For $\tau = 0$, the origin is an invariant curve whose Lyapunov exponent can be easily computed using (14):

$$\Lambda = \frac{1}{2\pi}\int_0^{2\pi} \ln|\alpha\cos\theta|\, d\theta = \ln|\alpha| - \ln 2.$$

Hence, for $\alpha < 2$, the invariant curve $x = 0$ is attracting. It can also be seen that, when α increases and crosses the critical value $\alpha = 2$, a SNA seems to branch off from the origin. Let us now focus on the case $\alpha = 3$ (the origin is a repellor). If we choose, for instance, $\tau = 0.5$, it is easy to see by direct simulation that (42) seems to have an attracting (and smooth) invariant curve. If we decrease τ, we see that the curve seems to fractalize and, for $\tau = 0$ it looks like an SNA. The Lyapunov exponent in all this process is negative (and far from 0). We believe that this corresponds to a torus collision, although the repelling torus does not exists (see Section 3.3) until the collision takes place. Therefore, this is not the scenario considered in this paper.

4.3 Quasi-periodically forced logistic map

Consider the two-parameter family of maps $f_{\alpha,\varepsilon} : \mathbb{R} \times \mathbb{T}^1 \mapsto \mathbb{R}$ defined by

$$f_{\alpha,\varepsilon}(x, \theta) = \alpha(1 + \varepsilon \cos(\theta))x(1 - x). \tag{43}$$

The corresponding dynamical system is

$$\left.\begin{array}{rcl} \bar{x} & = & f_{\alpha,\varepsilon}(x, \theta), \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\} \tag{44}$$

where we select $\omega = \pi(\sqrt{5} - 1)$. This map has been studied numerically in several papers (see, for instance, [PNR01] and references therein).

The reducibility of an invariant curve of this map can be discussed in a very simple way, by using the results in Section 2.1. If $x = u(\theta)$ denotes a continuous invariant curve of (44), its linear normal behaviour is given by

$$\left.\begin{array}{rcl} \bar{h} & = & D_x f_{\alpha,\varepsilon}(u(\theta), \theta)h = \alpha(1 + \varepsilon \cos\theta)(1 - 2u(\theta))h, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\}$$

where h denotes an infinitesimal displacement from the curve. Now let us focus on (the zeroes of) the expression $a(\theta) = (1+\varepsilon \cos\theta)(1-2u(\theta))$. It is clear that $|\varepsilon| \geq 1$ or $u(\theta_0) = \frac{1}{2}$ for some θ_0 imply non reducibility. On the other hand, if $|\varepsilon| < 1$, $u(\theta) \neq \frac{1}{2}$ (for all θ) and $u(\theta)$ is of class C^∞, Corollary 2.1 shows the reducibility of the curve. Note that the value $x = \frac{1}{2}$ is the critical point of the map (43).

The Lyapunov exponent of a curve $x = u(\theta)$ is

$$\Lambda(u) = \frac{1}{2\pi}\int_0^{2\pi} \ln|(1 + \varepsilon \cos\theta)(1 - 2u(\theta))|\,d\theta + \ln|\alpha|.$$

We will first consider the bifurcations of the invariant curve $u(\theta) \equiv 0$ for $\alpha > 0$. In this case, the Lyapunov exponent can be computed explicitly for all the values of ε. If we denote this exponent by $\Lambda_0(\alpha, \varepsilon)$, we have

$$\Lambda_0(\alpha, \varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} \ln|1 + \varepsilon\cos\theta|\,d\theta + \ln|\alpha| = \ln\left|\frac{1 + \sqrt{1 - \varepsilon^2}}{2}\right| + \ln|\alpha|.$$

This expression can be rewritten as

$$\Lambda_0(\alpha, \varepsilon) = \left\{\begin{array}{ll} \ln\left[\dfrac{1 + \sqrt{1 - \varepsilon^2}}{2}\right] + \ln|\alpha| & \text{if} \quad |\varepsilon| \leq 1 \\ \\ \ln\left|\dfrac{\varepsilon}{2}\right| + \ln|\alpha| & \text{if} \quad |\varepsilon| \geq 1 \end{array}\right.$$

Note that, for all α, $\Lambda_0(\alpha, \varepsilon)$ is continuous for $\varepsilon \in \mathbb{R}$ and real analytic for $\varepsilon \in \mathbb{R} \setminus \{\pm 1\}$.

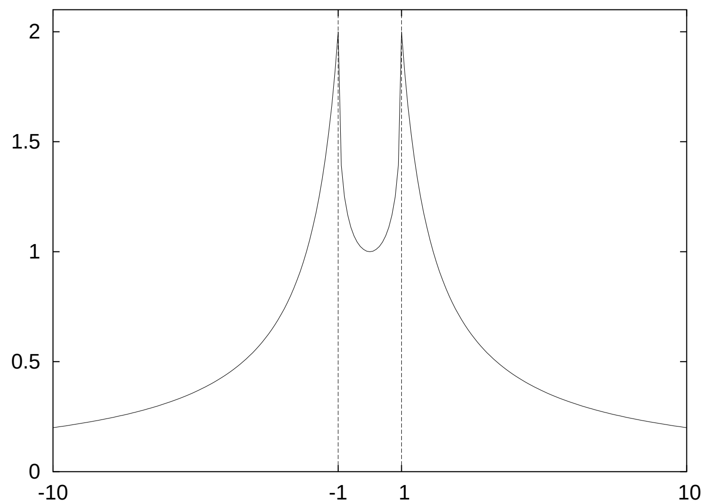

Figure 4: Bifurcation curve in the plane (ε, α) for the origin of the quasi-periodically forced logistic map. The region $|\varepsilon| < 1$ corresponds to reducible cases.

4.3.1 Bifurcations of $x = 0$. Reducible case

This is the simplest situation. As before, we focus on the curve $x(\theta) \equiv 0$ for $\alpha > 0$. Hence, to have reducibility, we need the condition $|\varepsilon| < 1$. Moreover, the reduced system

$$\left.\begin{array}{rcl} \bar{y} & = & by, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\}$$

has $b = \exp\left[\Lambda_0(\alpha, \varepsilon)\right] > 0$. Therefore,

$$b = \frac{\alpha}{2}\left(1 + \sqrt{1 - \varepsilon^2}\right). \tag{45}$$

The changes of stability correspond to the value $b = 1$, that is,

$$\alpha = \frac{2}{1 + \sqrt{1 - \varepsilon^2}}, \qquad |\varepsilon| \leq 1.$$

Note that, for $|\varepsilon| \leq 1$, $\alpha \in [1, 2]$. The graph is displayed in Figure 4.

4.3.2 Bifurcations of $x = 0$. Non reducible case

Now we consider the case $|\varepsilon| \geq 1$. The changes of stability take place when the parameters (α, ε) cross the curve $\Lambda_0(\alpha, \varepsilon) = 0$, that can be easily rewritten as

$$\alpha = \frac{2}{\varepsilon}, \qquad |\varepsilon| \geq 1.$$

The graph is contained in Figure 4.

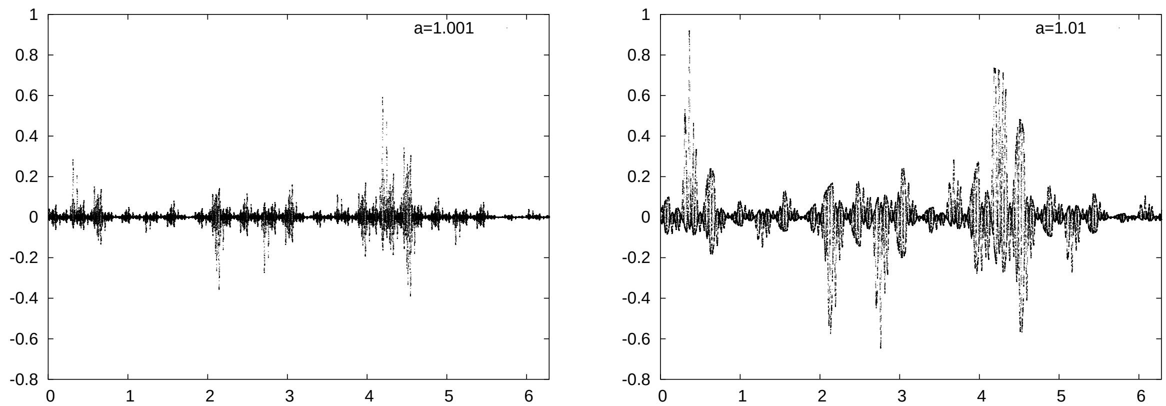

Figure 5: Attracting invariant set for the quasi-periodically forced logistic map with $\varepsilon = 2$, $\alpha = 1.001$ (left) and $\alpha = 1.01$ (right). The horizontal and vertical axis are θ and x, respectively.

To see the kind of bifurcation that occurs in this case, let us fix $\varepsilon = 2$. For $\alpha < 1$, the origin is an attracting curve. When α increases and crosses the critical value $\alpha = 1$, the origin changes its stability and a new attracting set bifurcates from the origin. We have drawn this set in Figure 5, for $\alpha = 1.001$ and $\alpha = 1.01$ (the corresponding Lyapunov exponents are -0.002852 and -0.020462).

Note that, if $|\varepsilon| > 1$, the quasi-periodically forced logistic map cannot have continuous invariant curve other than $x = 0$. This is because the zeros of the coefficient $1 + \varepsilon \cos\theta$ combined with the invariance imply a dense set of zeros for any invariant curve. Therefore, we think that these invariants sets are SNAs.

4.3.3 Fractalization of an invariant curve

We consider the case $\varepsilon = \frac{1}{2}$ and $\alpha > 0$, and we start focusing on the solution $x(\theta) \equiv 0$. From (45) we obtain that $x = 0$ is stable for $\alpha < \alpha_0 \equiv \frac{4}{2+\sqrt{3}} \approx 1.0717967697$ and unstable for $\alpha > \alpha_0$. As the origin can be seen as a reducible invariant curve, this bifurcation is standard in the sense that a stable (and reducible) invariant curve is born (see Figure 6, upper left, for $\alpha = 1.3$), at the same time that the origin becomes unstable. When α reaches a critical value $\alpha_1 \approx 1.65$, the curve crosses the line $x = \frac{1}{2}$ and then it becomes non reducible (see Figure 6, upper, right). If the value of α is increased, the curve becomes more irregular (see Figure 6, bottom).

Figure 7 shows the evolution of the Lyapunov exponent for $\varepsilon = \frac{1}{2}$ and α ranging between 0.5 and 2.7. The graphic clearly displays the change of stability of the origin when $\alpha = \alpha_0 = \frac{4}{2+\sqrt{3}}$. When $\alpha > \alpha_0$, the Lyapunov exponent starts decreasing until α reaches the value α_1 where the reducibility of the curve is lost. As predicted by Theorem 2.1, the derivative of the Lyapunov exponent goes to $-\infty$ when $\alpha \to \alpha_1^-$ and to a finite value when $\alpha \to \alpha_1^+$. When $\alpha > \alpha_1$, the Lyapunov exponent of the curve first approaches 0 and then it starts oscillating. Every time the curve has a new intersection with $x = \frac{1}{2}$, Theorem 2.1 implies that the derivative of the Lyapunov exponent must go to $-\infty$ and

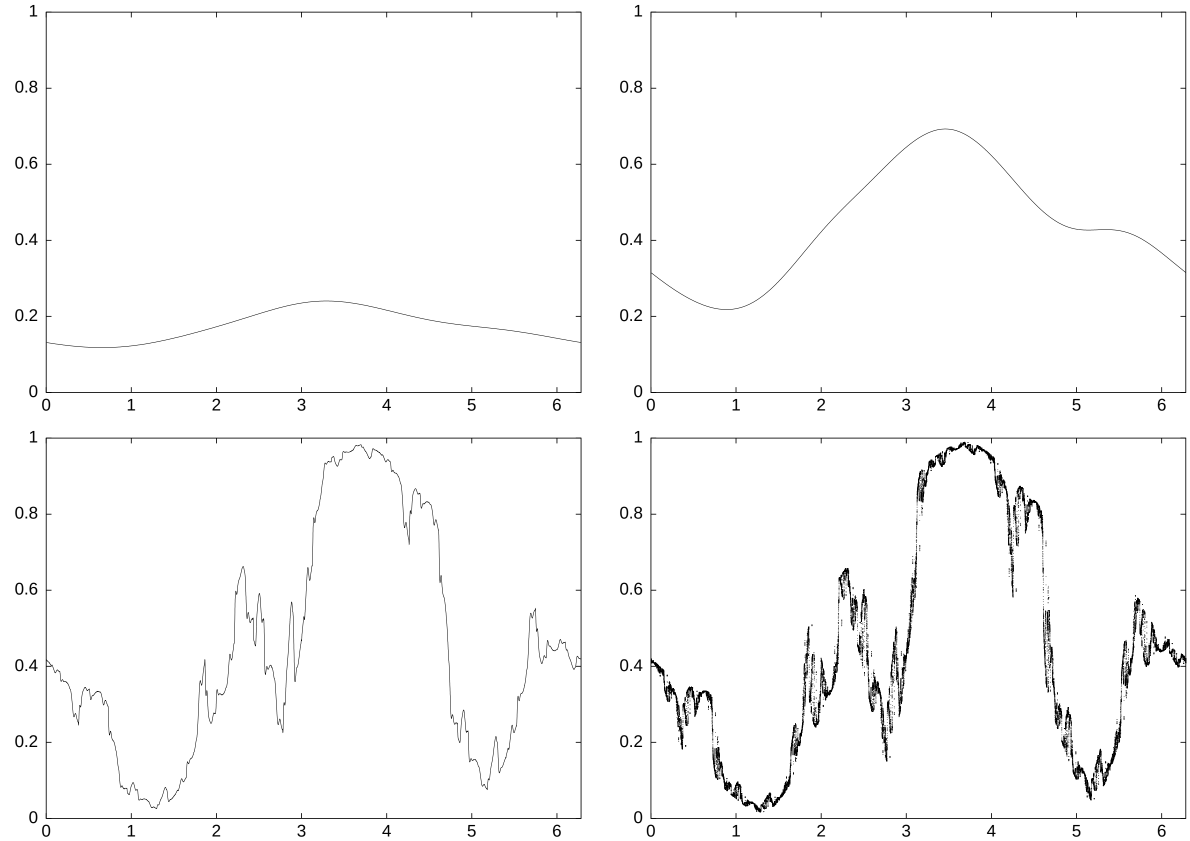

Figure 6: Attracting sets for the quasi-periodically forced logistic map, for $\varepsilon = \frac{1}{2}$. The horizontal axis refers to θ and the vertical axis refers to x. The values of α are 1.3, 2.0, 2.65 and 2.665.

then jump to a finite value. This is also seen in Figure 7.

We have also proved that if a C^r invariant curve is attracting, it can be locally continued with respect to the parameter α. Therefore, as the Lyapunov exponent seems to be always negative and that there is no evidence of a torus collision taking place, we believe that this attracting set is not an SNA but simply a smooth curve.

To give more numerical evidence that these "irregular" attracting sets are smooth curves, let us consider the following dynamical system,

$$\left.\begin{array}{rcl} \bar{x} & = & f(x,\theta), \\ \bar{y} & = & D_x f(x,\theta) y + D_\theta f(x,\theta), \\ \bar{\theta} & = & \theta + \omega. \end{array}\right\} \tag{46}$$

Note that, if $x = x(\theta)$ is a smooth invariant curve of (1), then $(x, y) = (x(\theta), x'(\theta))$ is an invariant curve of (46). This curve is attracting set of (46) iff $x = x(\theta)$ is an attracting set of (1). Now we will repeat the computations of the attracting sets of Figure 7 but on the system (46), to estimate the shape of the derivative of the curve, if there is one. In all the cases we will use the initial condition $y_0 = 1$ for the second equation in (46).

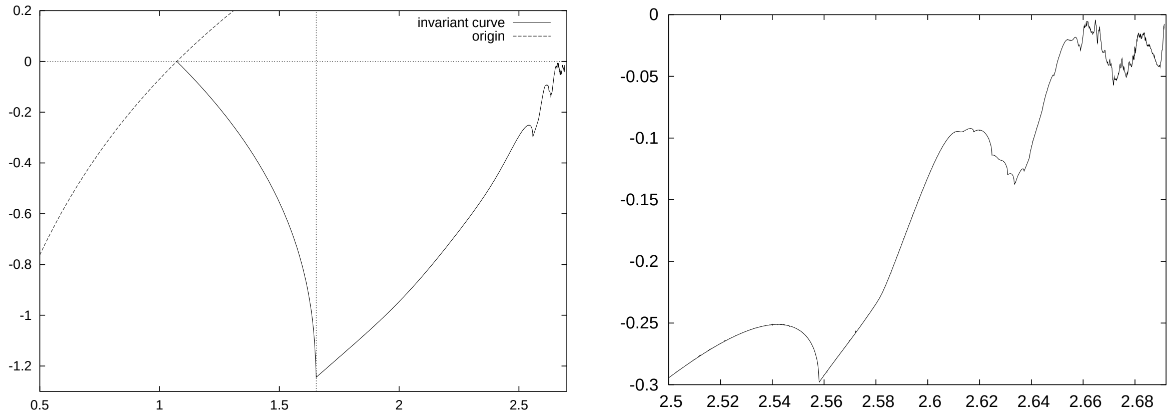

Figure 7: Horizontal axis: α; vertical axis: Lyapunov exponent for the quasi-periodically forced logistic map with $\varepsilon = \frac{1}{2}$. Left: the dashed line corresponds to the origin and the continuous line to the attracting invariant curve that bifurcates from the origin; the place where this curve looses the reducibility is marked with a vertical dotted line. Right: Magnification of the left plot.

The results are shown in Figure 8, for the same parameters values as in Figure 6. In the last case, $\alpha = 2.665$, we have used a logarithmic scale for $|y|$ to show the huge variation of the derivatives. Note that, if $x(\theta)$ is a smooth curve for the last case $\alpha = 2.665$, then $y = x'(\theta)$ must have a lot of zeros. As we only display the values $|x'(\theta)|$ on a finite mesh –of 10^5 points– we should only expect to "catch" values of $|x'|$ close to zero but positive.

To check whether the attractor for $\alpha = 2.665$ is a curve or not, we have performed several magnifications. If the attracting set is a curve, the values of y in (46) once we are on the attracting set can be used to estimate the maximum of the absolute value of the derivative. This quantity gives the amount of magnification needed to see the attractor as a smooth curve. After a transient of 10^6 iterates, we take the maximum of the derivative for 10^7 extra iterates, to obtain a value of -6.9×10^9 near $\theta_0 = 0.43748252111775532$. This process is very sensitive to roundoff error, especially from the modulus 2π needed for the variable θ (we will come back to this point later on). Therefore, different runs in different computers may give different values, but in all our tests the maximum of the derivative is of the order of 10^{10}. In particular, these estimates imply that to resolve a neighborhood of θ_0 we need magnifications of the order of 10^{10}, at least. Of course, we can magnify other parts of the curve but we have selected the point –of a sequence of 10^7 iterates– where the derivative is larger.

Hence, we will take the mesh $\theta_j = \theta_0 + \frac{j}{m}10^{-10}$ for j ranging from $-m$ to m. We have used several values of m between 100 and 1000. Then, we have computed the values $\hat{\theta}_j = \theta_j - n\omega (\operatorname{mod} 2\pi)$ for a large n (the concrete values are specified below) and we have iterated forward the points $\theta = \hat{\theta}_j$, $x = 0.4$, n times, to obtain the values $\tilde{\theta}_j$. These values should coincide with the initial values θ_j but, due to the roundoff errors (mainly in the operation $\operatorname{mod} 2\pi$) they are slightly different. For instance, for $n = 10^5$, the differences $\theta_j - \tilde{\theta}_j$ are close to 2.5×10^{-12}. To be sure that the results do not depend on the roundoff errors, we have repeated these computations with quadruple precision (we have used

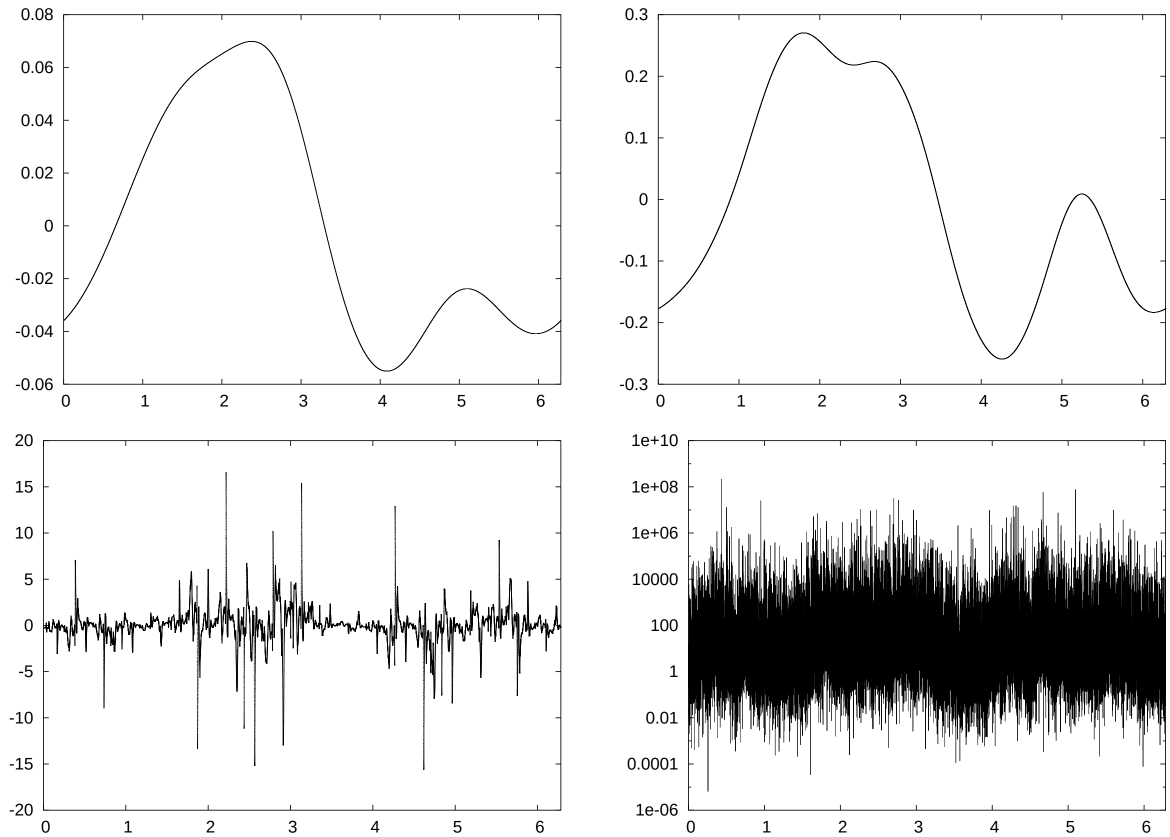

Figure 8: Attracting sets for the variational flow of the quasi-periodically forced logistic map for the attracting sets shown in Figure 6. The values of α are 1.3, 2.0, 2.65 and 2.665. The horizontal axis refers to θ and the vertical axis refers to y (see (46)). In the last plot we show $|y|$ in a log scale. See the text for details.

the library [BHJ$^+$]). Now, for $n = 10^5$ the differences $\theta_j - \tilde{\theta}_j$ are close to 1.7×10^{-23}. These results are shown in Figure 9 (left), where we have displayed the index j vs. the corresponding value of x. Note the differences between double and quadruple precision, and that the attractor looks like a clean smooth curve. The attractor for the equation of the derivative is shown in Figure 9 (right), and it also looks like a smooth curve.

Finally, to estimate the effect of the transient in these computations, we have repeated them for $n = 2 \times 10^5$ with no visible differences in the plots. We have also performed this zoom for other values of θ_0 with similar results.

One can argue that $\alpha = 2.665$ is still too small and that the SNA appears for a larger value. Then, given a larger α one can use the same process we have used here with extended precision to resolve the curve. We note that, for the fractalization scenario presented in Section 3, it is possible to select values of α for which the necessary amount of magnification is outside of the reach of present computers.

Our conclusion is that, although it is possible to obtain evidence of the existence of an attracting invariant curve by means of purely numerical methods, one has to be much more careful when dealing with SNAs.

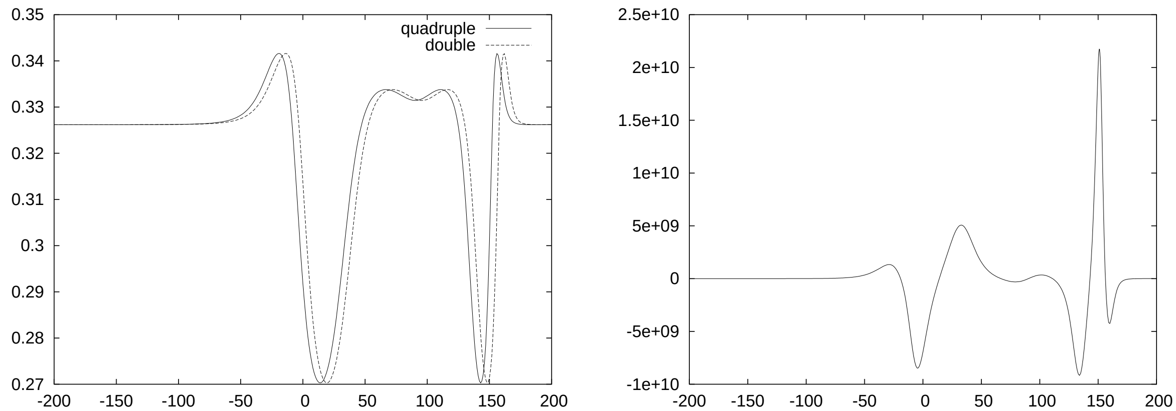

Figure 9: Zoom of the attracting set of the logistic map near $\theta_0 = 0.43748252111775532$, for $\alpha = 2.665$. The horizontal axis shows the value j corresponding to the angle $\theta_0 + \frac{j}{m}10^{-10}$ for $m = 200$. Left: Attracting set computed in double and quadruple precision. Right: Derivative of the left plot, estimated using the second equation in (46). See the text for details.

5 Final remarks

In this paper we have considered bifurcations of attracting curves of quasi-periodically forced 1-D systems from the point of view of the Implicit Function Theorem (IFT). We have shown that a failure of the IFT due to a null spectral value which is not an eigenvalue can result in a fractalization phenomena.

It is well known that if the null spectral value is an eigenvalue, the corresponding eigenvector (eigenfuntion in our case) is the linear approximation to the centre manifold at the critical point, which contains the relevant information for the bifurcation. The centre manifold can be seen as the minimal submanifold where the bifurcation takes place. Therefore, the usual procedure in this situation is to lower the dimensions of the problem by restricting the problem to this manifold.

However, when the null spectral value is not an eigenvalue, the situation changes completely. As there is no eigenvector, we cannot claim that the bifurcation is going to take place in a given low dimensional submanifold. At this point we recall that, if we are working in the space C^r, $r > 0$, the spectrum does not change if we replace r by r', $0 \le r' < r$. This implies that we do not have centre manifold for this bifurcation even in the space of continuous functions. Moreover, it is known ([HL05d]) that, if we work in the (larger) space

$$B = \{\beta : \mathbb{T}^1 \to \mathbb{R} \text{ such that } \sup_{\theta\in\mathbb{T}^1} |\beta(\theta)| < +\infty\},$$

endowed with the sup norm (we stress that functions in B do not need to be continuous or even measurable), then the spectrum consists of eigenvalues. This implies that, in principle, a natural space to study this bifurcation is B. Therefore, one should expect a complete loss of regularity when approaching such bifurcation, resulting in the fractalization phenomena that we have discussed here.

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