Parking a spacecraft near an asteroid pair

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The purpose of this paper is to study the dynamics of a spacecraft moving in the field of a binary asteroid. The asteroid pair is modeled as a rigid body and a sphere moving in a plane, while the spacecraft moves in their field in space. This model is interesting because it is one of the simplest that captures the coupling between rotational and translational dynamics. By assuming that the binary dynamics is in a relative equilibrium, a restricted model for the spacecraft in orbit about them is constructed that also includes the direct effect of the Sun on the spacecraft dynamics. The standard Restricted Three Body Problem (RTBP) is used as starting point for the analysis of the spacecraft motion. In particular, how the triangular points of the RTBP are modified through perturbations is investigated by taking into account two sorts of perturbations, namely that one of the primaries is no longer a point mass but is an extended rigid body, and second, taking into account the effect of orbiting the Sun. The stable zones near the modified triangular equilibrium points of the binary and a normal form of the Hamiltonian around them are used to compute stable periodic and quasi-periodic orbits for the spacecraft, which enable it to observe the asteroid pair while the binary orbits around the Sun.

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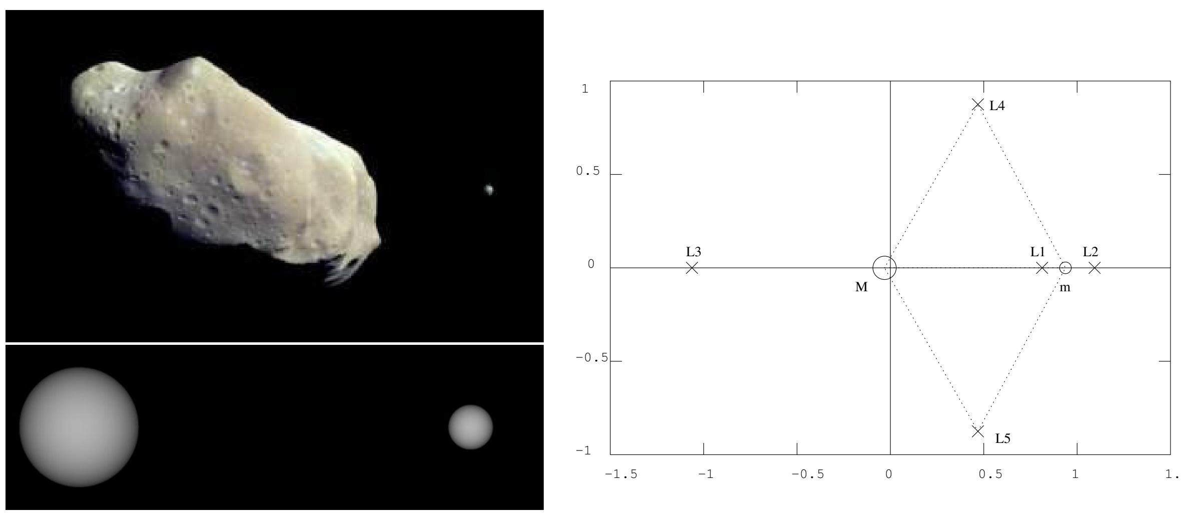
I. Introduction

In the last decade, asteroid satellites and double asteroids have had a prominent place in the discovery of new Solar System objects[1] . Since 1993, when the Galileo spacecraft discovered *Dactyl*, the first natural satellite of an asteroid ever found—in this case of the asteroid *(243)-Ida* (see Figure 1(a))—over 50 binary asteroids have been discovered and the interest in studying asteroid pairs has grown significantly[1] . Examples of these discoveries include the pairs *(22)-Kalliope and Linus* and *(45)-Eugenia and Petit-Prince*, in the main belt; or the Near–Earth Asteroids *(3671)-Dionysus and S/1997* and *2000 DP107 and S/2000* (see Figure 1(a)).



(a) (b)

Figure 1. (a) Top: Ida and Dactyl (source: JPL). Bottom: A schematic diagram representing the sizes of the 2000 DP107 components and their separation, drawn to scale (source: J. L. Margot, Caltech). (b) The five equilibrium points of the RTBP.

Therefore, the study of spacecraft motion about an asteroid pair is an extremely relevant topic for future missions to asteroids as 16% of Near-Earth Asteroids (NEA) are thought to be binaries[2] . Binaries can be used as real-life laboratories to test rigid-body gravitational dynamics[1] . For instance, an important question is to find stable zones and orbits near the asteroid pair where a spacecraft can "park" to carry out measurements and observations of the binary as the pair orbits around the Sun. See Figure 2(a).

In solving this problem, we draw on some basic facts of the circular Restricted Three Body Problem[3] (RTBP). The RTBP describes the motion of a massless particle under the gravitational attraction of two massive bodies (primaries) which presumably revolve in circular orbits around their common center of masses. Usually, the system is studied in a rotating frame (synodical), where the two massive bodies are fixed on the x axis. The details on how to derive the corresponding equations of motion can be found, for instance, in Ref. 3.

As it is well known, the RTBP has five equilibrium points (in synodical coordinates). Three of them are on the x axis and they are known as collinear points or L_1, L_2 and L_3. The remaining two form equilateral triangles with the massive bodies and are known as triangular points or L_4 and L_5. See Figure 1(b).

While the collinear points are always unstable, the stability of the triangular points depends on the value of $\mu = m/(M+m)$, where m and M are the masses of the primaries. If $\mu < \mu_R$, where $\mu_R = \frac{1}{2}(1 - \sqrt{23/27})$ (known as the *Routh critical value*), the triangular points are spectrally stable, otherwise they are unstable. These "equilateral" points are of interest to us because their stability properties will vary due to the rigid body effects.

The main aim of this paper is thus to study how these triangular equilibria are perturbed in two important ways:

(1) When one of the primaries is not a point mass any more but an extended rigid body.

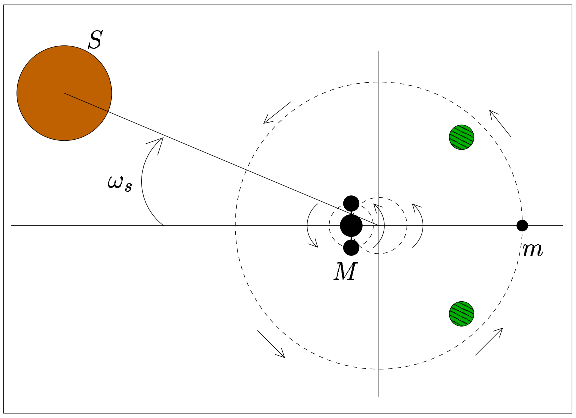
(2) When the effect of orbiting the Sun is also considered.

Under this situation, the coupling between the dynamics of the relative translational motion of the two bodies and the rigid body rotation has to be taken into account. Furthermore, a time-dependency will appear due to the perturbation of the Sun.

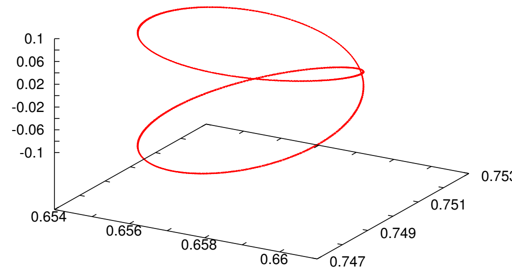
As for the collinear unstable points, they are also worth studying since we known, via *Genesis Discovery Mission*, for example, that it can be cheap to park a spacecraft near them by using the so-called Center Manifold[4] . But, the study of the collinear points under the perturbation of rigid body effects will be taken up in a future work.

We use a simple model for the asteroid pair, a planar system made up of a rigid body and a sphere. This model is known as "sphere restriction" of the Full Two Body Problem[5] (F2BP). The F2BP considers the gravitational interaction between two general distributed bodies. See Ref. 6, for a formal definition of the problem and some initial stability studies. See Refs. 7–9 for more studies on stability of the F2BP, including the sphere restriction case.

To model the spacecraft motion, we assume the binary to be in a relative equilibrium and we also consider the direct effect of the Sun on the spacecraft. Other studies in the literature[10,11] use the approach taken in Hill's problem, in which the Sun is taken at infinity, to tackle the influence of the perturbation of the Sun.



(a)



(b)

Figure 2. (a) Schematic diagram showing the two stable zones for the spacecraft (gray circles) to observe the binary (M and m) as the asteroid pair orbits around the Sun (S). (b) A high inclination observation orbit for the spacecraft.

The basic techniques used in the present paper are taken from geometric mechanics and dynamical systems theory. The use of Hamiltonian reduction methods allows us to reduce the dimension of the problem and *Normal Form* techniques are central to our numerical explorations. The software is "handcrafted" (adapted from the programs in Refs. 12 and 13; see also Ref. 14) and uses an algebraic manipulator to obtain high-order expansions (Taylor and Fourier-Taylor series) of the functions involved in the computations. These high-order expansions cannot be achieved with a commercial-type manipulator and are important, for instance, to obtain relatively high inclination observation orbits for the spacecraft. See Figure 2(b).

The paper is organized as follows: In Section II, we derive the reduced equations for the asteroid pair via reduction theory and make a preliminary study of this reduced model. Abelian reduction theory used in obtaining the binary model is reviewed in the Appendix. In Section III, we construct the models for the spacecraft motion based on the a particular relative equilibrium solution of the asteroid pair. Only the direct effect of the Sun on the spacecraft (and not on the asteroid pair) is considered in the modeling. In Section IV, we study the dynamics of these models in the vicinity of the stable triangular points and use this study to find stable periodic and quasi-periodic orbits suitable for parking the spacecraft while it observes the binary. Finally, in Section V, the conclusions and future work are presented.

II. A Model of the Asteroid Pair

To obtain a model of asteroid pair, we apply the general Abelian reduction process (the details of which are recalled in the Appendix) to the particular case of the planar F2BP with the sphere restriction.

A. Equations of motion for the binary

Consider the mechanical system of a rigid body and a sphere in space, but that are moving in a plane, as in Figure 3.

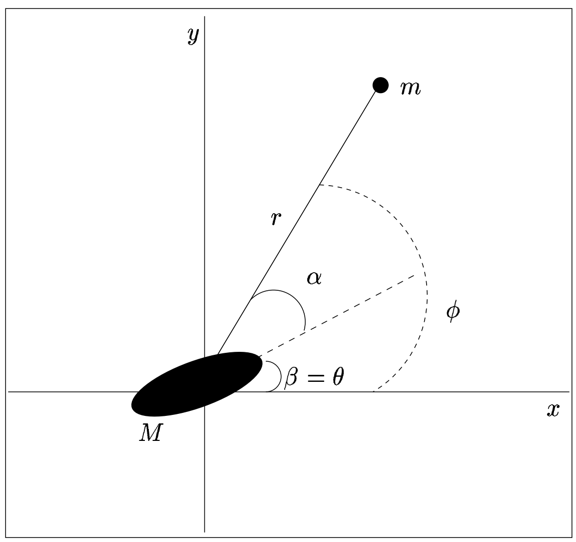


Figure 3. Gravitational interaction of a rigid body and a sphere in the plane.

1. Reduction of the translational symmetry

Relative to a given inertial reference frame, the kinetic energy of the system is

$$K = \frac{1}{2}m\|\dot{\mathbf{r}}\|^2 + \frac{1}{2}M\|\dot{\mathbf{R}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2,$$

where $\mathbf{r}$ and $\mathbf{R}$ are the positions of the sphere's center and the barycenter of the rigid body, m and M are the masses of the sphere and the rigid body, I_{zz} is the inertia tensor of the rigid body and the angle θ is as shown in Figure 3.

It is straightforward to perform a first reduction using the invariance of the system under translations. This can be achieved by using the fact that at the system's center of mass, $m\mathbf{r} + M\mathbf{R} = 0$. After defining $\mathbf{q} = \mathbf{r} - \mathbf{R}$, which is the relative position of the sphere with respect to the rigid body, one gets $\mathbf{r} = \frac{M}{m+M}\mathbf{q}$ and $\mathbf{R} = \frac{-m}{m+M}\mathbf{q}$, and the kinetic energy can be re-written as

$$K = \frac{1}{2}\frac{mM}{m+M}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2.$$

Furthermore, if the unit of mass is defined such that $\frac{mM}{m+M} = 1$, the unit of length is taken to be the longest axis of inertia of the rigid body, and the unit of time is chosen such that $G(m+M) = 1$, then the kinetic energy can be simplified as

$$K = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2.$$

Notice that the configuration space Q of this reduced system is the planar Euclidean group $SE(2)$. Its Lagrangian is of the type kinetic minus potential and can be locally written as

$$L(\mathbf{q}, \theta, \dot{\mathbf{q}}, \dot{\theta}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \frac{1}{2}I_{zz}\dot{\theta}^2 - V(\mathbf{q}, \theta). \tag{1}$$

From the Lagrangian, we obtain the momenta conjugate to the variables $(\mathbf{q}, \theta)$ via the Legendre transformation: $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}$, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_{zz}\dot{\theta}$. Thus, its corresponding Hamiltonian is

$$H = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{2I_{zz}}p_\theta^2 + V(\mathbf{q}, \theta). \tag{2}$$

This system still has an overall rotational symmetry, with respect to which we will reduce next.

2. Reduction of the rotational symmetry

We first perform two preliminary (canonical) changes of variables that will simplify the action of the symmetry group $G = S^1$ on the configuration space $Q = SE(2)$. The first change is the introduction of polar coordinates:

$$q_x = r\cos\phi, \qquad p_x = p_r\cos\phi - \frac{p_\phi}{r}\sin\phi,$$
$$q_y = r\sin\phi, \qquad p_y = p_r\sin\phi + \frac{p_\phi}{r}\cos\phi.$$

The second one is the use of the relative angles

$$\alpha = \phi - \theta, \qquad p_\alpha = p_\phi,$$
$$\beta = \theta, \qquad p_\beta = p_\phi + p_\theta,$$

as shown in Figure 3. These changes are the first steps in rewriting the equations of the system using the *body frame* of the rigid body. After performing these changes, the Lagrangian becomes

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\alpha}^2 + \frac{1}{2}(r^2 + I_{zz})\dot{\beta}^2 + r^2\dot{\alpha}\dot{\beta} - V(r,\alpha).$$

Note that the potential does not depend on the "orientation" angle θ due to its invariance under rotations. Also notice that the action of the symmetry group G on the (r,α,β) variables is trivial:

$$\Phi_\varphi(r,\alpha,\beta) = (r,\alpha,\beta+\varphi).$$

Moreover, the Hamiltonian in these new coordinates is given by:

$$H = \frac{1}{2}p_r^2 + \left(\frac{1}{2r^2} + \frac{1}{2I_{zz}}\right)p_\alpha^2 + \frac{1}{2I_{zz}}p_\beta^2 - \frac{1}{I_{zz}}p_\alpha p_\beta + V(r,\alpha), \tag{3}$$

where $p_\alpha = r^2\dot{\alpha} + r^2\dot{\beta}$ and $p_\beta = r^2\dot{\alpha} + (r^2 + I_{zz})\dot{\beta}$. Notice that β is a cyclic variable of the Hamiltonian (3) and therefore, its conjugate momentum p_β is conserved.

To perform the reduction on the Hamiltonian side, we apply the theory reviewed in the Appendix (for more details, see Refs. 15 and 16). The *momentum map* is given by

$$\mathbf{J}(r,\alpha,\beta,p_r,p_\alpha,p_\beta) = p_\beta.$$

which corresponds to the angular momentum of the system in the new coordinates. The *locked inertia tensor* is $\mathbb{I}(r,\alpha,\beta) = r^2 + I_{zz}$, which is the instantaneous inertia tensor when the relative motion of the two body is locked. The *mechanical connection* is the 1-form given by

$$\mathcal{A}(r,\alpha,\beta) = \frac{r^2}{r^2+I_{zz}}d\alpha + d\beta.$$

For a fixed $p_\beta = \gamma$, we can perform the momentum shift from $\mathbf{J}^{-1}(\gamma)$ to $\mathbf{J}^{-1}(0)$ by means of

$$\tilde{p}_r = p_r, \qquad \tilde{p}_\alpha = p_\alpha - \frac{\gamma r^2}{r^2+I_{zz}}, \qquad \tilde{p}_\beta = 0.$$

The reduced Hamiltonian in $\mathbf{J}^{-1}(0)/S^1$ has only two degrees of freedom

$$H = \frac{1}{2}\tilde{p}_r^2 + \frac{1}{2}\left(\frac{1}{r^2}+\frac{1}{I_{zz}}\right)\tilde{p}_\alpha^2 + V(r,\alpha) + \frac{\gamma^2}{2(r^2+I_{zz})} \tag{4}$$

with the non-canonical reduced symplectic form given by

$$\omega_\gamma = dr\wedge d\tilde{p}_r + d\alpha\wedge d\tilde{p}_\alpha - \frac{2\gamma I_{zz} r}{(r^2+I_{zz})^2}dr\wedge d\alpha. \tag{5}$$

Finally, the reduced Hamiltonian equations can be easily derived from the Hamiltonian (4) and the symplectic form (5). It is a system with two degrees of freedom, which describes the motion of the sphere in the body frame of the rigid body

$$\begin{aligned}
\dot{r} &= \tilde{p}_r,\\
\dot{\alpha} &= \left(\frac{1}{r^2}+\frac{1}{I_{zz}}\right)\tilde{p}_\alpha,\\
\dot{\tilde{p}}_r &= \frac{\tilde{p}_\alpha^2}{r^3} - \frac{\partial V(r,\alpha)}{\partial r} + \frac{\gamma^2 r}{(r^2+I_{zz})^2} + \frac{2\gamma I_{zz} r}{(r^2+I_{zz})^2}\left(\frac{1}{r^2}+\frac{1}{I_{zz}}\right)\tilde{p}_\alpha,\\
\dot{\tilde{p}}_\alpha &= -\frac{\partial V(r,\alpha)}{\partial\alpha} - \frac{2\gamma I_{zz} r}{(r^2+I_{zz})^2}\tilde{p}_r,
\end{aligned}$$

where $V(r,\alpha)$ is the potential of the rigid body in the body-frame.

3. Simple potential of the rigid body

For simplicity, we approximate the potential of the rigid body by the gravitational potential of three masses stuck together with two massless rigid rods. The two external masses will be assumed to be identical (see Figure 4).

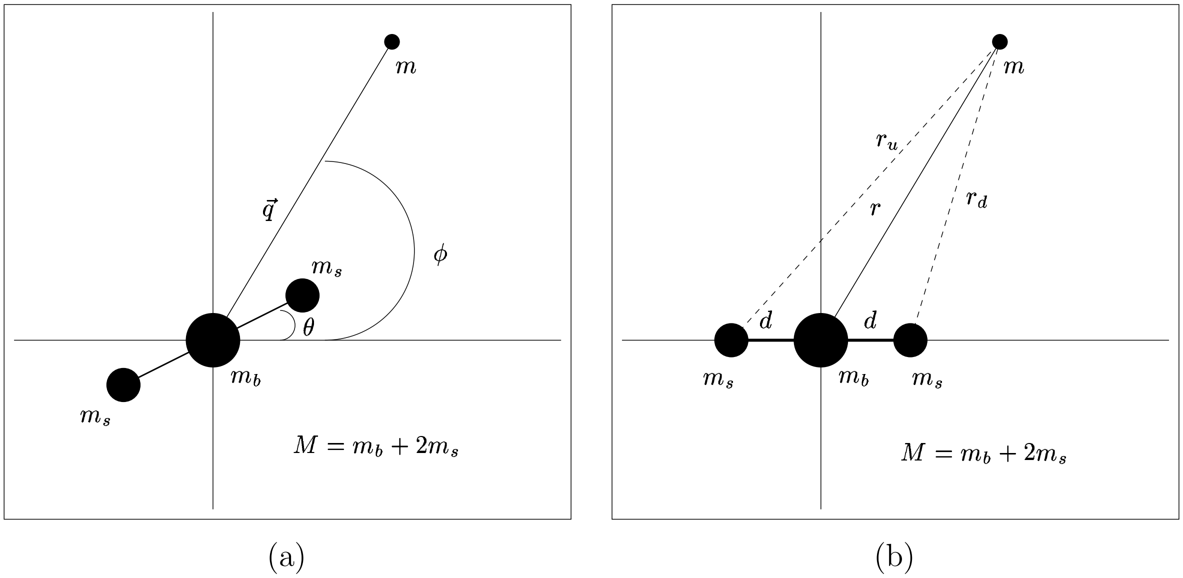


Figure 4. Simple model for the potential of the rigid body. (a) Unreduced system. (b) System in the body-frame.

Following the results obtained in the last section, the Hamiltonian with this potential is

$$H = \frac{1}{2}\tilde{p}_r^2 + \frac{1}{2}\left(\frac{r^2 + I_{zz}}{r^2 I_{zz}}\right)\tilde{p}_\alpha^2 + V_\gamma(r,\alpha), \tag{6}$$

where

$$V_\gamma = -\frac{1-2\mu}{r} - \mu\left(\frac{1}{r_u} + \frac{1}{r_d}\right) + \frac{\gamma^2}{2(r^2 + I_{zz})}.$$

Here, (see Figure 4(b)), $\gamma \in \mathbb{R}$, $\mu = \frac{m_s}{m_b + 2m_s}$, $\nu = \frac{m}{m+M}$, $r_u^2 = r^2 + 2dr\cos\alpha + d^2$ and $r_d^2 = r^2 - 2dr\cos\alpha + d^2$. The moment of inertia of the system is $I_{zz} = \frac{\mu}{2\nu}$. Sometimes it will be useful to use I_{zz} as a parameter instead of ν. The Hamiltonian equations can be readily derived from Hamiltonian (6) and the symplectic form (5).

B. Relative equilibria for the binary

The relative equilibria of the asteroid pair in the reduced system can be obtained from the Hamiltonian (6). They satisfy the following equations:

$$\tilde{p}_r = \tilde{p}_\alpha = 0, \qquad \frac{\partial V_\gamma}{\partial r} = \frac{\partial V_\gamma}{\partial \alpha} = 0.$$

After some straightforward computations, we obtain

$$
\begin{aligned}
p_r = 0, \quad p_\alpha = \frac{\gamma r^2}{r^2 + I_{zz}}, \quad \mu d r \sin\alpha \left(\frac{1}{r_d^3} - \frac{1}{r_u^3} \right) &= 0, \\
\frac{1-2\mu}{r^2} - \frac{\gamma^2 r}{(r^2 + I_{zz})^2} + \mu \left(\frac{r + d\cos\alpha}{r_u^3} - \frac{r - d\cos\alpha}{r_d^3} \right) &= 0.
\end{aligned}
$$

The third equation gives us the solution for the orientation angle α and the last one the distance r. From the first two equations, we can compute the momenta once the relative positions are known. There are two types of solutions, depending on the value of the orientation angle:

- *Collinear configurations*, with $\sin\alpha = 0$, $\alpha \in \{0, \pi\}$;
- *T-configurations*, with $r_d = r_u$, which is equivalent to $\alpha = \pm\dfrac{\pi}{2}$.

In this paper, we do not intend to do a general study of the stability of these relative equilibria. Instead, we will focus our attention only on the cases when the rigid body is "big" ($\nu << 1$), and when the binary is in a stable configuration which enables us to study the motion of a spacecraft near the pair. Our preliminary numerical experiments show that the collinear configurations are likely to be unstable (see Ref. 9 for a related problem). These results motivate one to study the T-configuration in more detail. By computing the eigenvalues of the linearized vector field at the relative equilibria for a range of parameter values, we can establish the spectral stability of these relative equilibria as parameter values vary.

An example of the results is shown in Figure 5. Here, we have fixed the relative mass of the binary ν to two typical values[10] : $\nu = 10^{-2}$ and $\nu = 10^{-3}$. For instance, the mass parameter for some known binaries are: $\nu \sim 10^{-4}$ for Ida, $\nu \sim 10^{-3}$ for Kalliope, $\nu \sim 2\times 10^{-3}$ for Eugenia, $\nu \sim 2 \times 10^{-2}$ for Dionysus and $\nu \sim 5 \times 10^{-2}$ for 2000 DP107.

In Figure 5, the parameter values corresponding to spectrally stable relative equilibria are colored in gray. The values of the parameters studied here are $\mu \in (0, 0.2)$ and $\omega \equiv \dfrac{\gamma}{I_{zz}} \in (0, 5)$. The moment of inertia of the rigid body is taken as $I_{zz} = \frac{\mu}{\nu}$.

In Section III, we use the result summarized in Figure 5 to choose concrete values for the parameters such that the T-configuration is spectrally stable.

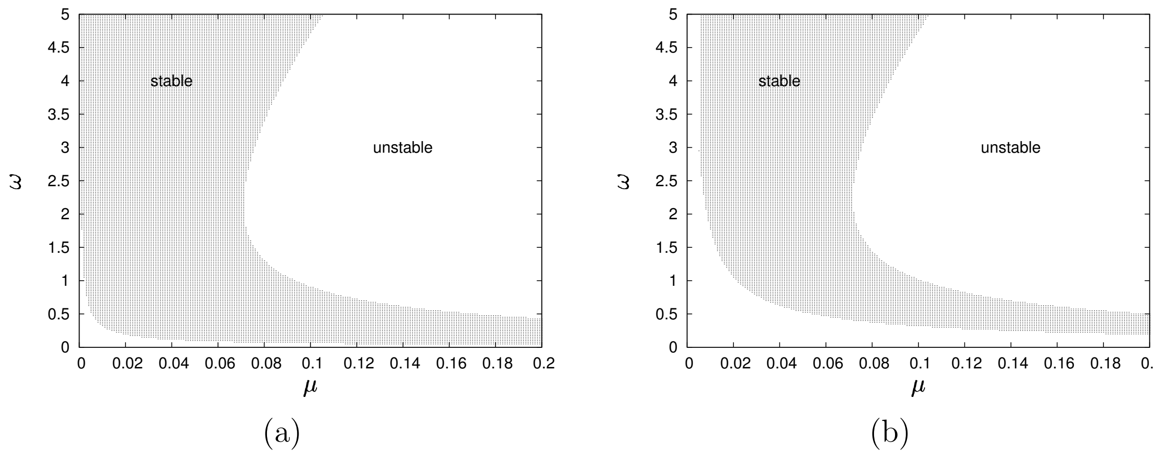


Figure 5. Spectral stability for the T-configuration (gray zone). (a) $\nu = 10^{-3}$. (b) $\nu = 10^{-2}$.

1. Reconstruction of the Relative Equilibria

Since we are interested in visualizing the relative equilibria in the initial configuration space, we need to reconstruct the dynamics from the reduced coordinates. In our case, it is not difficult to see that the reconstruction equations for the group variables are given by

$$\dot{\theta} = \frac{p_\theta}{I_{zz}}, \qquad p_\theta = \gamma - p_\alpha. \tag{7}$$

If a solution in the reduced space $(r(t), \alpha(t), p_r(t), p_\alpha(t))$ is known, one may integrate equation (7) to obtain the evolution of the orientation angle of the rigid body $\theta(t)$.

For instance, if the reduced system is in one of the relative equilibria described above, the conjugate momenta of the orientation angle variable is constant; that is, $p_\theta =$ constant, and the equation for the attitude (7) is trivial to integrate:

$$\dot{\theta} = \frac{p_\theta}{I_{zz}} \equiv \omega_L \text{ which implies that } \theta = \omega_L t + \theta_0.$$

Hence, in the unreduced system, the relative equilibria are periodic orbits of period $T_L = \frac{2\pi}{\omega_L}$. For example, for one of the T-configuration points ($r \equiv r_L$, $\alpha = \frac{\pi}{2}$), the solution for a rigid body and a sphere is a T-shaped object which is rotating uniformly at rate ω_L. See Figure 6(a).

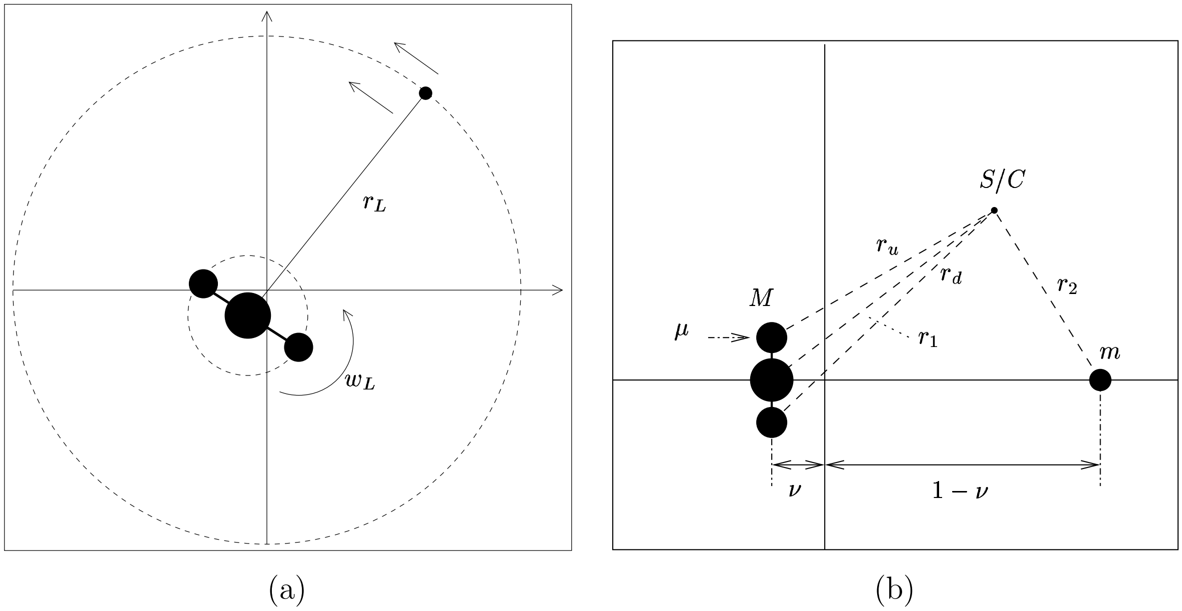


Figure 6. (a) Relative equilibria for the T-configuration visualized in the unreduced reference frame. The system is rotating uniformly with angular velocity ω_L. (b) Basic T-model for the spacecraft (S/C).

III. Models for the Motion of the Spacecraft

In this section, we will construct two models for the motion of a spacecraft near the asteroid pair. Let us assume that the binary is in a specific relative equilibrium with $\alpha = \pi/2$ and the system is rotating uniformly with frequency ω_L, as in Figure 6(a). In the first model, we will assume that the motion of the spacecraft is affected only by the gravitational interaction of the asteroid pair. In the second model, we will add the effect of the perturbation of the Sun on the equations of motion of the spacecraft.

A. Basic T-model

Suppose that $\mathbf{Q}_0$ and $\{\mathbf{Q}_u, \mathbf{Q}_d\}$ are, respectively, the position vectors of the central and the two external masses of the rigid body in an inertial reference frame centered at the system's barycenter. Let us also call $\mathbf{Q}_s$ the position of the sphere and $\mathbf{Q}$ the position of the spacecraft in the same frame.

In this inertial reference frame, the kinetic energy of the spacecraft is given by $K = \frac{1}{2}\|\dot{\mathbf{Q}}\|^2$, and thus the corresponding momenta can be defined as $\mathbf{P} = \dot{\mathbf{Q}}$. The equations of motion for the spacecraft can be written as

$$\dot{\mathbf{Q}} = \mathbf{P}, \qquad \dot{\mathbf{P}} = -\frac{\partial V}{\partial \mathbf{Q}},$$

where the potential is given by

$$V = -G\left(\frac{m_b}{\|\mathbf{Q}_{0p}\|} + \frac{m_s}{\|\mathbf{Q}_{up}\|} + \frac{m_s}{\|\mathbf{Q}_{dp}\|} + \frac{m}{\|\mathbf{Q}_{sp}\|}\right),$$

with $M = m_b + 2m_s$, $\nu = m/(m+M)$, and $\mathbf{Q}_{jp} = \mathbf{Q} - \mathbf{Q}_j$, for $j = 0, u, d, s$.

We now perform a rotation to fix the rigid body's longest principal axis orthogonal to the x-axis: $\mathbf{Q} = \mathcal{R}_{\theta_L}\mathbf{q}$, where $\mathbf{q} = (x, y, z)$ and $\mathcal{R}_{\theta_L}$ is the counterclockwise rotation of angle $\theta_L = \omega_L t + \theta_0$ in the xy plane,

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this *rotating frame*, the equations of motion for the spacecraft are:

$$\begin{aligned} \dot{x} &= p_x + \dot{\theta}_L y, & \dot{p_x} &= \dot{\theta}_L p_y - \frac{\partial V}{\partial x}, \\ \dot{y} &= p_y - \dot{\theta}_L x, & \dot{p_y} &= -\dot{\theta}_L p_x - \frac{\partial V}{\partial y}, \\ \dot{z} &= p_z, & \dot{p_z} &= -\frac{\partial V}{\partial z}, \end{aligned}$$

where

$$V = -G\left(\frac{m_b}{\|\mathbf{q}_{0p}\|} + \frac{m_u}{\|\mathbf{q}_{up}\|} + \frac{m_d}{\|\mathbf{q}_{dp}\|} + \frac{m}{\|\mathbf{q}_{sp}\|}\right).$$

Here, $\dot{\theta}_L = \omega_L$ and $\mathbf{q}_{jp} = \mathbf{q} - \mathbf{q}_j$, for $j = 0, u, d, s$. Note that $\mathbf{q}_j$ (position vectors of the masses in the rigid body frame) are known from the T-configuration relative equilibria: $\mathbf{q}_0 = (-\nu r_L, 0, 0)$, $\mathbf{q}_u = (-\nu r_L, 1/2, 0)$, $\mathbf{q}_d = (-\nu r_L, -1/2, 0)$ and $\mathbf{q}_s = ((1-\nu) r_L, 0, 0)$.

These equations are Hamiltonian (in a canonical way) with the following Hamiltonian function:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega_L\left(y p_x - x p_y\right) + V(x, y, z).$$

Let us redefine non dimensional units for the spacecraft model as follows: take the new unit of length to be the distance between the center of masses of the rigid body and the sphere, the unit of time to be such that $\omega_L = 1$, so that the asteroid pair does a complete revolution in 2π units of time, and the unit of mass such that $GmM = 1$.

Then, the Hamiltonian for the motion of the spacecraft can be written as an $O(\mu)$-perturbation of the RTBP with mass-ratio ν

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + (yp_x - xp_y) + V(x, y, z), \tag{8}$$

where

$$V = -\frac{(1-\nu)(1-2\mu)}{r_1} - \frac{\nu}{r_2} - \mu(1-\nu)\left(\frac{1}{r_u} + \frac{1}{r_d}\right).$$

Here, $r_1^2 = (x+\nu)^2 + y^2 + z^2$, $r_2^2 = (x-(1-\nu))^2 + y^2 + z^2$, $r_u^2 = (x+\nu)^2 + (y-d)^2 + z^2$, $r_d^2 = (x+\nu)^2 + (y+d)^2 + z^2$ and $d = \dfrac{1}{2r_L}$, as in Figure 6(b).

B. T-Model: Perturbation of the Sun

We now take into consideration the direct effect of Sun's perturbation on the spacecraft. We assume, as a first approximation, that the uniform rotation of the binary is not affected by the Sun and that the center of masses of the binary is also rotating uniformly around the Sun with a rate denoted by ω_s, as in Figure 7(b). This idea is similar to the construction of the well-known Bicircular Problem[17] (BCP) that has been used to model some restricted four-body problems in the Solar System[18,19] .

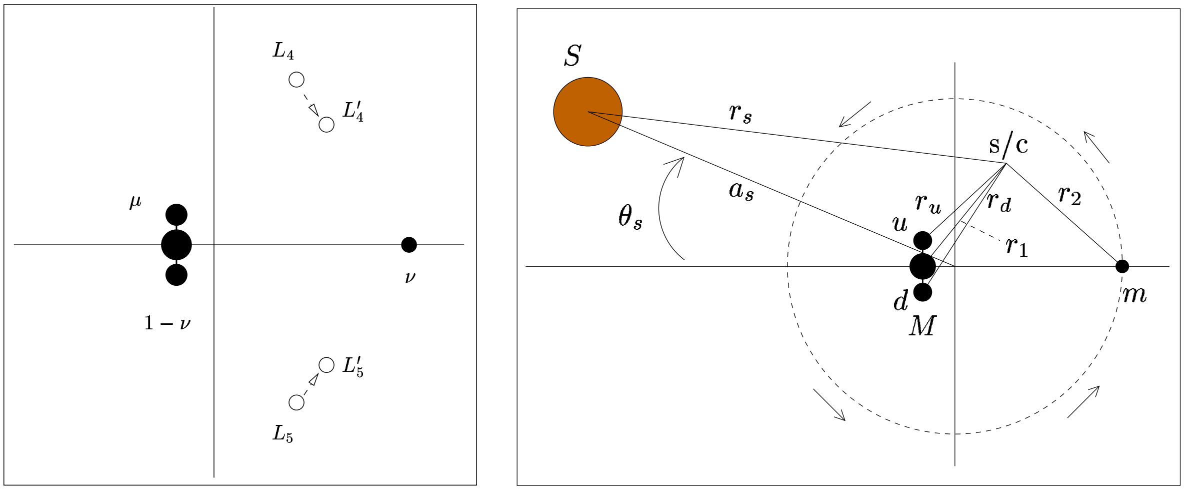
Let us sketch the construction of the model. We start in inertial coordinates, where we denote $\mathbf{Q}$ the position vector of the spacecraft measured from the Sun and $\{\mathbf{Q}_1, \mathbf{Q}_u, \mathbf{Q}_d, \mathbf{Q}_2\}$ the ones corresponding, respectively, to the center, upper and lower mass of the rigid body, and to the sphere. In these coordinates, the equations of motion for the spacecraft are

$$\ddot{\mathbf{Q}} = -\frac{\partial V}{\partial \mathbf{Q}},$$

where the Newtonian potential is:

$$V = -\frac{m_1}{\|\mathbf{Q}-\mathbf{Q}_1\|} - \frac{m_u}{\|\mathbf{Q}-\mathbf{Q}_u\|} - \frac{m_d}{\|\mathbf{Q}-\mathbf{Q}_d\|} - \frac{m_2}{\|\mathbf{Q}-\mathbf{Q}_2\|} - \frac{m_s}{\|\mathbf{Q}\|}.$$

We perform two changes of variables to write the equations in the so-called synodical



(a) (b)

Figure 7. (a) Schematic diagram showing the triangular equilibria of the RTBP persist when one of the primaries is not a point mass but an extended rigid body. (b) Perturbation of the Sun on the T-model.

coordinates relative to the binary. The first one is a translation from the Sun to the center of masses of the asteroid pair: $\mathbf{Q} = \mathbf{Q}_{CM_T} + \mathbf{R}$, where $\mathbf{Q}_{CM_T} = (a_s \cos\bar{\theta}_s, a_s \sin\bar{\theta}_s, 0)$, $\bar{\theta}_s = n_s t + \bar{\theta}_s^0$ and $\mathbf{R} = (X, Y, Z)$ is the position of the spacecraft from the center of masses of the T-model (CM_T). After this (time-dependent) translation, the equations for the spacecraft are

$$\ddot{\mathbf{R}} = a_s n_s^2 (\cos n_s t, \sin n_s t, 0) - \frac{\partial V}{\partial \mathbf{R}}.$$

The second change of variables is a (time-dependent) rotation which fixes the binary to the x-axis:

$$\begin{aligned} X &= x\cos t - y \sin t, \\ Y &= x \sin t + y \cos t, \\ Z &= z. \end{aligned}$$

From here, it is easy to write:

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= \ddot{X}\cos t + \ddot{Y}\sin t, \\ \ddot{y} + 2\dot{x} - y &= \ddot{Y}\cos t - \ddot{X}\sin t. \end{aligned}$$

Recall that, in the relative equilibria of the binary, $(X_1, Y_1) = (-\nu \cos t, -\nu \sin t)$, $(X_2, Y_2) = ((1-\nu)\cos t, (1-\nu)\sin t)$ and similar expressions for m_u and m_d hold. If we denote $\mathbf{r} = (x, y, z)$ to be the position of the spacecraft in these rotating coordinates, we obtain

$$
\begin{aligned}
\ddot{x} &= 2\dot{y} + x + \frac{m_s}{a_s^2}\cos\theta_s - m_1 \frac{x+\nu}{\|\mathbf{r}-\mathbf{r}_1\|} - m_2 \frac{x-(1-\nu)}{\|\mathbf{r}-\mathbf{r}_2\|} \\
&\quad - m_u \frac{x+\nu}{\|\mathbf{r}-\mathbf{r}_u\|} - m_d \frac{x+\nu}{\|\mathbf{r}-\mathbf{r}_d\|} - m_s \frac{x + a_s\cos\theta_s}{\|\mathbf{r}-\mathbf{r}_s\|}, \\
\ddot{y} &= -2\dot{x} + y - \frac{m_s}{a_s^2}\sin\theta_s - m_1 \frac{y}{\|\mathbf{r}-\mathbf{r}_1\|} - m_2 \frac{y}{\|\mathbf{r}-\mathbf{r}_2\|} \\
&\quad - m_u \frac{y-d}{\|\mathbf{r}-\mathbf{r}_u\|} - m_d \frac{y+d}{\|\mathbf{r}-\mathbf{r}_d\|} - m_s \frac{y - a_s\sin\theta_s}{\|\mathbf{r}-\mathbf{r}_s\|}, \\
\ddot{z} &= -m_1 \frac{z}{\|\mathbf{r}-\mathbf{r}_1\|} - m_2 \frac{z}{\|\mathbf{r}-\mathbf{r}_2\|} \\
&\quad - m_u \frac{z}{\|\mathbf{r}-\mathbf{r}_u\|} - m_d \frac{z}{\|\mathbf{r}-\mathbf{r}_d\|} - m_s \frac{z}{\|\mathbf{r}-\mathbf{r}_s\|},
\end{aligned}
$$

where $\theta_s = t - \bar{\theta}_s = t - n_s t - \pi$ and the third Kepler law has been used in the computations.

Finally, defining the momenta in the usual way via the Legendre transform, we get

$$
\begin{aligned}
\dot{x} &= p_x + y, \\
\dot{y} &= p_y - x, \\
\dot{z} &= p_z,
\end{aligned}
$$

the equations of motion are Hamiltonian and the Hamiltonian function can be written as a periodic perturbation of the T-model:

$$
H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + (y p_x - x p_y) - \frac{m_s}{a_s^2}(x\cos\theta_s - y\sin\theta_s) + V(x, y, z, \theta_s), \tag{9}
$$

where

$$
V = -\frac{(1-\nu)(1-2\mu)}{r_1} - \frac{\nu}{r_2} - \mu(1-\nu)\left(\frac{1}{r_u} + \frac{1}{r_d}\right) - \frac{m_s}{r_s}. \tag{10}
$$

Here, $r_s^2 = (x + a_s\cos\theta_s)^2 + (y - a_s\sin\theta_s)^2 + z^2$ and $\theta_s = \omega_s t + \theta_s^0$.

IV. Nonlinear Dynamics near the Perturbed Lagrange Points

Recall that the triangular libration points of the RTBP are spectrally stable if the mass parameter is smaller than the Routh critical value. In this case, the nonlinear dynamics around these points can be studied by means of Normal Form techniques.[12,20–22] In this section, we study the effect of the rigid body and the Sun near the triangular points and use similar techniques to describe the nonlinear dynamics near them. Furthermore, many quasi-periodic trajectories can be found near these fixed points that will be useful for spacecraft orbiting around the asteroid pair.

To choose the parameters for the models developed in Section III, we use the stability results of the asteroid pair relative equilibria described in Subsection IIB. In particular, we choose the parameters such that the binary is in a stable T-configuration. The mass ratio between the sphere and the rigid body is taken as a typical value for certain type of binaries, $\nu = m/(m+M) = 0.001$. The grade of mass dispersion μ of the rigid body is chosen to be relatively small, $\mu = 0.02$, and its moment of inertia is taken as $I_{zz} = 20$. The angular momentum is also taken moderately small, $\gamma = 4$, so that the T-configuration is spectrally stable with $\omega = 0.2$ as its corresponding frequency. See Figure 5(a). With these parameters, the T-configuration solution is found at $r_L = 5.07830172847938$ times the largest dimension of the rigid body. Also, the selection of these particular values ensures that the perturbed triangular points will be spectrally stable.

When the perturbation of the Sun is taken into account, we assume that the binary is in the main asteroid belt ($a_s \approx 3$ A.U.) and that its total mass is that of a medium/large size asteroid (10^{17} kg). This gives us the rest of the parameters for the second model in adimensional units: $a_s = 1.5 \times 10^6$ and $m_s = 10^{13}$. The relative frequency of the Sun, ω_s, can be easily obtained from the third Kepler law: $\omega_s = 0.998278674068352$.

Using these parameter values to make the actual computations, we proceed to make a local study of the dynamics for the spacecraft near the Lagrangian stable regions, knowing that the qualitative results will be valid for a wide range of parameters.

The Implicit Function Theorem shows that, if the perturbations are small and under some non-resonance conditions, the RTBP triangular points persist in the basic T-model and are replaced by stable periodic orbits after taking into consideration of the Sun's perturbation. See Figure 7(a).

In this paper, we focus on the L_4 case, although the same results can be obtained for

L_5. The new fixed point that plays the role of L_4 in the T-model will be called L_4' and the periodic orbit that has the same period as the Sun's perturbation $T_s = \frac{2\pi}{w_s}$ will be named $PO(L_4')$.

A. Study of the dynamics at L_4' and $PO(L_4')$

It is not difficult to compute the eigenvalues of the linearized vector field at L_4' or the Floquet multipliers of the periodic orbit $PO(L_4')$. For the actual example, they correspond to elliptic objects and are displayed in Table 1.

Thus, the T-model system is elliptic at L_4' and around $PO(L_4')$ and we can study the nonlinear dynamics around these objects by constructing a high-order Normal Form around the fixed point L_4' and around the periodic orbit $PO(L_4')$.

We briefly describe the main points of the procedure as follows—the reader can consult Refs. 12 and 13 for details.

1. Translate the origin of coordinates to L_4' or $PO(L_4')$. In the second case, the translation depends periodically on time.

2. Construct the quadratic Normal Form using the real Jordan form of the linearization of the vector field in the autonomous case and the Floquet Theorem in the time-periodic case.

3. Perform an expansion of the Hamiltonian in a Taylor series (Fourier–Taylor series in the time-periodic case) up to degree N.

4. Construct a high-order Normal Form with a Lie series method[12] .

Next, we explain these main steps in more detail.

1. Quadratic Normal Form

We focus on the time-periodic case (for the autonomous system, the quadratic normal form corresponds to the real Jordan form of the linear differential equations): After having moved the origin to the periodic orbit $PO(L_4')$ (this translation is T_s-periodic), the linear part of the vector field is of the form $\dot{u} = L(t)u$, where $L(t)$ is a 6×6 T_s-periodic real matrix.

The Floquet theorem ensures the existence of a linear T_s-periodic change of variables $v = C(t)u$ such that the linear part of the vector field reduces to a linear system with constant coefficients $\dot{v} = \Lambda v$, where Λ is a 6×6 real constant matrix.

It is possible[13] to choose the change of variables such that $C(t)$ is symplectic (the method requires this condition because the changes are made directly on the Hamiltonian function) and the matrix Λ takes the form

$$\Lambda = \left(\begin{array}{c|c} 0_3 & \Omega \\ \hline -\Omega & 0_3 \end{array}\right),$$

where 0_3 is the 3×3 zero matrix and $\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$ is the diagonal matrix containing the frequencies w_j corresponding to the three normal modes of the periodic orbit (recall that the periodic orbit $PO(L_4')$ is elliptic).

Implementing these changes of variables, the Normal Form up to degree 2 only contains monomials of order 2. The order 0 is irrelevant in the equations of motion and the order 1 terms are eliminated because the origin is a fixed point after the translation. So, the normal form up to degree two is a quadratic form given by

$$H_2 = \omega_1 \frac{x_1^2 + y_1^2}{2} + \omega_2 \frac{x_2^2 + y_2^2}{2} + \omega_3 \frac{x_3^2 + y_2^3}{2}.$$

Finally, it is convenient, before starting the high-order Normal Form, to implement a complexification of the variables

$$x_j = \frac{q_j + ip_j}{\sqrt{2}}, \quad y_j = \frac{iq_j + p_j}{\sqrt{2}}, \quad j = 1, 2, 3\,,$$

which rewrites the quadratic part of the Hamiltonian in the following form:

$$H_2 = i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + i\omega_3 q_3 p_3,$$

where the values for the frequencies w_j can be found in Table 1.

	L_4'	$PO(L_4')$
ω_1	-0.10702011607983	-0.10702058242758
ω_2	0.99366842989866	0.99366615570514
ω_3	1.00058470215019	1.00058692342681

Table 1. Normal frequencies for the linear oscillators around the elliptic fixed point L_4' and periodic orbit $PO(L_4')$.

2. High-order Normal Form

Prior to the construction of the high-order Normal Form, we expand the Hamiltonian in Fourier-Taylor series (Taylor series in the autonomous case) and insert the previous linear changes to this expansion to obtain

$$H = i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + i\omega_3 q_3 p_3 + \sum_{j \geq 3} H_j(q, p, \theta_s),$$

which will be the starting input object of the following construction. For details on how to expand the current type of potential functions, see Ref. 13.

To build the Normal Form of order higher than 2, we use the Lie series method (see Ref. 23 or Ref. 24, for an introduction) implemented as in Ref. 12. We use a hand-made software that contains an algebraic manipulator that is able to deal with the Taylor and Fourier–Taylor series appearing in the computations. We sketch one step of the process for the time-periodic case. For the autonomous case, just skip the dependence with time.

Let us assume that the Hamiltonian is already in normal form up to degree $r - 1$:

$$H = \omega_s p_{\theta_s} + H_2^{(n)}(qp) + \sum_{j=4,\ j=\dot{2}}^{r-1} H_j^{(n)}(qp) + H_r(q, p, \theta_s) + H_{r+1}(q, p, \theta_s) + \cdots$$

where $H_r(q, p, \theta_s) = \sum_{|k|=r} h_r^k(\theta_s) q^{k^1} p^{k^2}$, $\theta_s = \omega_s t + \theta_0$ and $k = (k^1, k^2) \in \mathbb{Z}^3 \times \mathbb{Z}^3$. The extra term $\omega_s p_{\theta_s}$ has been introduced to autonomize the Hamiltonian and p_{θ_s} is the momenta conjugated to the θ_s variable.

We are interested in making a change of variables such that the homogeneous polynomial $H_r(q, p, \theta_s)$ takes a form that is as simple as possible. In particular, we want this change to make the monomials contained in it autonomous. It is easy to see that the canonical change given by the following generating function satisfies these requirements:

$$G_r = G_r(q, p, \theta_s) = \sum_{\substack{|k|=r \\ k^1 \neq k^2}} \frac{-h_r^k(\theta_s)}{\langle \omega, k^2 - k^1 \rangle} q^{k^1} p^{k^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product.

The new Hamiltonian obtained with this change of variables is obtained with the Lie series[23, 24] :

$$H' = H + \{H, G_r\} + \frac{1}{2!}\{\{H, G_r\}, G_r\} + \frac{1}{3!}\{\{\{H, G_r\}, G_r\}, G_r\} + \cdots$$

and can be written as:

$$H' = \omega_s p_{\theta_s} + H_2^{(n)}(qp) + \sum_{j=4,\, j=\dot{2}}^{r-1} H_j^{(n)}(qp) + H_r^{(n)}(qp) + H'_{r+1}(q, p, \theta_s) + \cdots$$

We iterate this process and perform all the changes up to a high-order N. More concretely, we use $N = 32$ in the autonomous case and $N = 24$ in the time-periodic one. These particular choices are due to RAM memory limitations, but they are already high enough for our purposes. We will give more details in the next section.

Finally, we write the Hamiltonian in action-angle variables (I, φ), by defining $I_j = i q_j p_j$, $j = 1, 2, 3$,

$$H = \mathcal{N}(I) + \mathcal{R}(I, \varphi, \theta_s)\,, \qquad \mathcal{N}(I) = \sum_{|k|=1}^{N/2} h_k I_1^{k_1} I_2^{k_2} I_3^{k_3}, \tag{11}$$

where the term $\mathcal{R}(I, \varphi, \theta_s)$ is the remaining part of the Hamiltonian that has not been transformed to Normal Form, and thus it still depends on the angles and time. This term is of, at least, order $N + 1$.

We can now assume that the dynamics in a vicinity of L'_4 and $PO(L'_4)$ is given by the Normal Form Hamiltonian $\mathcal{N}(I)$. The coefficients of the Normal Forms up to order 6 in the square root of the actions are given in Table 2.

Since the Normal Form only depends on actions, it is trivially integrable. All motions in a (small) vicinity of L'_4 and $PO(L'_4)$ are periodic or quasi-periodic. They take place on invariant tori of dimensions 1, 2 and 3 (autonomous case) or 2, 3 and 4 (periodic case). See Figures 8 and 9 for some examples.

3. Test and validity of the Normal Form approximation

Along with the Normal Form computation, we have constructed the transformations (using the generating function written as a Fourier–Taylor expansion) that send points from the Normal Form space to the initial one and vice versa. These changes of variables not only provide a way to visualize the dynamics in the initial coordinates, but also they are very

k_1	k_2	k_3	h_k (*T-model*)	h_k (*T-model + Sun*)
1	0	0	-1.07020116079827e-01	-1.07020582427575e-01
0	1	0	9.93668429898665e-01	9.93666155705138e-01
0	0	1	1.00058470215019e+00	1.00058692342681e+00
2	0	0	1.98209282337547e+00	1.98210606475071e+00
1	1	0	-8.41335633534186e-02	-8.41363103519259e-02
0	2	0	3.34451789294228e-02	3.34473273382156e-02
1	0	1	9.77435542208162e-02	9.77447151079432e-02
0	1	1	1.62579166640980e-02	1.62670378418912e-02
0	0	2	-7.22623320607685e-04	-7.23728487410444e-04
3	0	0	7.23164870868422e+01	7.23172271456241e+01
2	1	0	2.40637121303353e+01	2.40641088603377e+01
1	2	0	6.06695219686117e+00	6.06715074575221e+00
0	3	0	-9.09462225134519e-02	-9.09462282070405e-02
2	0	1	-3.51875180623665e+00	-3.51880095271078e+00
1	1	1	3.17368174336265e+00	3.17373202727457e+00
0	2	1	-1.72093135781978e-01	-1.72064978874555e-01
1	0	2	-1.02419399482606e-01	-1.02420550719694e-01
0	1	2	-6.60967536055335e-03	-6.64293506149859e-03
0	0	3	3.82965067009504e-05	3.82868741518696e-05

Table 2. Coefficients of the normal forms up to order 6 in the square root of the actions. The first three columns contain the exponents of the actions. The fourth column corresponds to the autonomous case and the fifth one to the time-perturbed model.

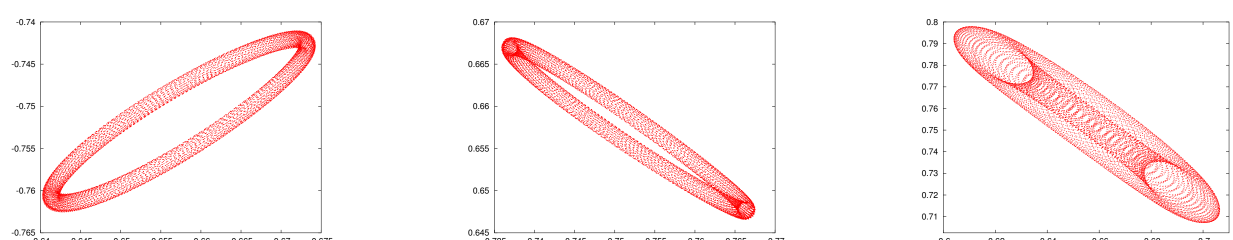


Figure 8. Examples of stable orbits near the Lagrangian zones of the asteroid pair I. Left/Center: Projection into the xp_x and yp_y planes, respectively, of a 3D torus. Right: Projection into the xy plane of 4D torus for the time-periodic case.

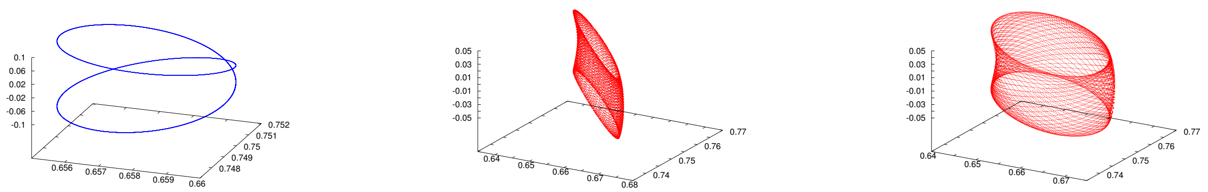


Figure 9. Examples of stable orbits near the Lagrangian zones of the asteroid pair II. Left: Periodic orbit in the autonomous case. Center/Right: Two examples of 3D tori for the time-periodic case.

helpful to test the programs.

In order to check the correctness of the computations, we proceed as follows (see Ref. 12 for a more detailed description of these tests): First, we take a fixed value for the actions $I_1 = I_2 = I_3 = \lambda$, with λ small, transform this point back to the initial coordinates and integrate it numerically using the vector field corresponding to (8) and (9) during a time span T. Let us call this final point $\mathbf{x}(\lambda)$. Then, we use the Normal Form (11) to integrate (this integration is a trivial tabulation) the same initial condition ($I_j = \lambda$) in the "Normal Form space" for the same interval of time T and transform back the final point of the integration to the initial coordinates. We call now this point $\mathbf{x}'(\lambda)$.

The norm of the difference between $\mathbf{x}(\lambda)$ and $\mathbf{x}'(\lambda)$ is an increasing function of λ and gives an estimation of the error that we are doing when approximating the dynamics of models (8) and (9) by the corresponding Normal Forms. Moreover, if we denote

$$e(\lambda) = \|\mathbf{x}(\lambda) - \mathbf{x}'(\lambda)\|_2, \tag{12}$$

the order of the error approximately behaves like

$$N \approx \frac{\ln\left(\dfrac{e(\lambda_1)}{e(\lambda_2)}\right)}{\ln\left(\dfrac{\lambda_1}{\lambda_2}\right)},$$

where N corresponds to the order of the Normal Form. This test is performed using different (small) values of λs and successfully passed by all our programs.

B. Spacecraft Parking Orbits

We can construct prescribed stable trajectories for the spacecraft near the elliptic objects L_4' and $PO(L_4')$ by using the dynamics given by the Normal Form (11). Some of these stable orbits are very interesting because we can use them to "park" the spacecraft to do observations of the binary as the pair orbits around the Sun.

Notice that, essentially, the I_1 and I_2 actions correspond to planar motion (in the xy plane) and I_3 to the vertical direction zp_z. This observation is not exact because the presence of nonlinear terms, but it will be very useful for some applications.

For instance, if we are interested in performing observations of the asteroid pair with relatively high inclinations, we can prescribe values for the initial conditions with $I_1 = I_2 = 0$ and $I_3 = \lambda_3$. The concrete value of λ_3 should be taken as large as possible. This is determined by asking that the Normal Form approximation error (12) is smaller than certain tolerance, $e(\lambda_3) < \text{Tol}$. In our computations, for instance, if we choose $\text{Tol} = 10^{-10}$, we can take $\lambda_3 = 0.02$.

With this particular choice of actions, we obtain trajectories for the spacecraft that correspond to a periodic orbit in the autonomous case and a 2-D invariant torus in the time-dependent model. These particular trajectories are shown in Figure 10. Notice that they are quite extended in the vertical direction.

Figure 10. Examples of relatively high inclination orbits suitable for binary observations. Left: Periodic orbit for the autonomous case. Right: 2D invariant torus for the time-periodic case.

V. Conclusions and Future Work

In this paper we have studied a simple model for the motion of the spacecraft near an asteroid pair. First, we have computed the relative equilibria for the Full 2 Body Problem (whose bodies are moving in a plane) and made a preliminary exploration of their stability. Then, we have modeled the motion of the spacecraft assuming that the asteroid pair remains in this relative equilibria while it evolves in a circular orbit around the Sun.

The Lagrangian points of the Restricted Three Body Problem have served as a starting point to find zones of practical stability suitable for placing the spacecraft. We have studied how these points are modified by the effects due to the fact that (1) one body is an extended body, and (2) the perturbation of the Sun. Moreover, we have used dynamical systems tools, such as the construction of a high-order Normal Form, to study the non-linear dynamics around these Lagrangian points. Finally, these accurate numerical methods have enabled us to construct periodic and quasi-periodic orbits very convenient for doing observations of the asteroid pair.

Future work will consider other interesting aspects of the problem. For instance, different solutions of the F2BP (i.e. not in relative equilibria) may bring us to quasi-periodic models[25] , where the primary rigid body rotates with a different angular velocity than the translational rotation of the second spherical body around the primary. These models can be studied for example with the methods developed in Ref. 14. We also plan to study more realistic potentials for the non-spherical body. Furthermore, the problem of two full bodies, without the sphere restriction, seems to be a challenging system to study.

As mentioned before, it is also of our interest to investigate rigid body perturbations on orbits in the Central Manifolds of the collinear points, as these orbits have already been used in real missions (b.g. *Genesis*).

In a current work in progress[26] , we analyze the problem of finding binary observation orbits for the spacecraft from another point of view. We use a completely different computational tool, namely the frequency map analysis[27] , to obtain the global dynamics of the stability region near the Lagrangian points of the Restricted Full 3 Body Problem. From this global picture, we are able to identify (almost) invariant tori and quasi-periodic trajectories on these invariant tori that are very convenient for "parking" the spacecraft on them, while the spacecraft is observing the asteroid pair.

Appendix: Abelian reduction. General setting

In this appendix, we review the reduction process for a system that is invariant under the abelian Lie group $SO(2)$. We perform the reduction on both sides of the problem, Lagrangian and Hamiltonian, and show that they are equivalent via the reduced Legendre transform.

Let us start by assuming that the configuration space Q can be written as a product of the circle S^1 and a manifold B called *shape space*, i.e., $Q = S^1 \times B$ and $q = (q^0, q^\alpha) = (\theta, r^\alpha)$, with $q \in Q$, $\theta \in S^1$ and $r^\alpha \in B \subseteq \mathbb{R}^n$.

Let us assume that the symmetry group $G = SO(2) = S^1$ acts trivially in the following way:

$$\begin{array}{rcl} \Phi : G \times Q & \longrightarrow & Q \\ (\varphi, q) & \longmapsto & \Phi_\varphi(q) = \Phi_\varphi(\theta, r^\alpha) = (\theta + \varphi, r^\alpha) \end{array}$$

where G is a Lie group with Lie algebra $\mathfrak{g} = \mathbb{R}$ and dual Lie algebra $\mathfrak{g}^* \cong \mathbb{R}$.

We also assume that the Lagrangian is of the type *kinetic* minus *potential*. It can be written, in a local trivialization of TQ, as follow:

$$L(q, \dot{q}) = K(q, \dot{q}) - V(q) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q).$$

where g_{ij} is a Riemannian metric and summations over $i, j = 0, 1, \ldots, n$ are understood. The corresponding Hamiltonian on T^*Q is given by

$$H(q, p) = K(\mathbb{F}L(q, \dot{q})) + V(q) = \frac{1}{2} g^{ij} p_i p_j + V(q),$$

where g^{ij} is the inverse of the metric g_{ij}, $(q, p) = \mathbb{F}L(q, \dot{q})$ is the Legendre transform of $(q, \dot{q})$ $((q^i, p_i) = (q^i, g_{ij}\dot{q}^j))$ and the symplectic form is canonical, i.e., $\Omega = dq^i \wedge dp_i$.

A. Lagrangian reduction

We start with the Lagrangian re-written in the following form:

$$L(r^\alpha, \dot{\theta}, \dot{r}^\alpha) = \frac{1}{2} g_{00} \dot{\theta}^2 + g_{0\alpha} \dot{\theta} \dot{r}^\alpha + \frac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V(r^\alpha).$$

Thus, θ is a cyclic variable and the corresponding conjugate momenta $p_\theta = g_{00}\dot{\theta} = \frac{\partial L}{\partial \dot{\theta}}$ is conserved. While the classical theory of Routh reduction is valid, we will use modern Routh reduction[28] that applies in a much more general framework.

The ingredients needed in the reduction process (see Ref. 28 for details) are:

- **Infinitesimal generator**: The infinitesimal generator corresponding to the group action can be computed as follow:

$$\xi_Q(\theta, r^\alpha) = \frac{d}{dt}\left(\exp(t\xi)\cdot(\theta, r^\alpha)\right)|_{t=0} = ((\theta, r^\alpha), (\xi, 0)).$$

- **Lagrangian momentum map**: The associated momentum map $\mathbf{J}_L : TQ \leftarrow \mathfrak{g}^*$. is given by

$$\mathbf{J}_L((q,\dot{q}))\cdot\xi_Q = \langle \mathbb{F}L(q,\dot{q}), \xi_Q(q)\rangle = \left\langle (g_{0j}\dot{q}^j, g_{\alpha j}\dot{q}^j), (\xi, 0)\right\rangle = g_{0j}\dot{q}^j\xi.$$

Thus, $\mathbf{J}_L(q,\dot{q}) = g_{00}\dot{\theta} + g_{0\alpha}\dot{r}^\alpha$.

- **Locked inertia tensor**: The locked inertia tensor is the instantaneous tensor of inertia when the relative motion of the two bodies is locked. If we denote $\langle\langle\cdot,\cdot\rangle\rangle$ the scalar product induced by the metric g_{ij}, the locked inertia tensor $\mathbb{I}(\theta, r^\alpha) : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given (locally) by

$$\langle \mathbb{I}(\theta, r^\alpha)\eta, \xi\rangle = \langle\langle ((\theta, r^\alpha), (\eta, 0)), ((\theta, r^\alpha), (\xi, 0))\rangle\rangle = g_{00}\eta\xi.$$

Then, $\mathbb{I}(\theta, r^\alpha) = g_{00}(r^\alpha)$.

- **Mechanical connection**: The connection $\mathcal{A} : TQ \longrightarrow \mathfrak{g}$ can be written (locally) as $\mathcal{A}(\theta, r^\alpha)(\theta, r^\alpha, \dot{\theta}, \dot{r}^\alpha) = \mathbb{I}^{-1}\mathbf{J}(\mathbb{F}L(\theta, r^\alpha, \dot{\theta}, \dot{r}^\alpha)) = g_{00}^{-1}g_{0j}\dot{q}^j$. Thus,

$$\mathcal{A}(\theta, r^\alpha)(\theta, r^\alpha, \dot{\theta}, \dot{r}^\alpha) = \dot{\theta} + g_{00}^{-1}g_{0\alpha}\dot{r}^\alpha.$$

From $\mathcal{A}$, we can obtain the related one-form: $\mathcal{A}(\theta, r^\alpha) = d\theta + A_\alpha dr^\alpha$, where $A_\alpha = g_{00}^{-1}g_{0\alpha}$, and the curvature $\mathcal{B} = d\mathcal{A} = B_{\alpha\beta}dr^\alpha \wedge dr^\beta$ has components given (locally) by $B_{\alpha\beta} = \left(\frac{\partial A_\alpha}{\partial r^\beta} - \frac{\partial A_\beta}{\partial r^\alpha}\right)$. For a given $\mu \in \mathfrak{g}^* \cong \mathbb{R}$, the mechanical connection on the fiber $Q \rightarrow Q/G$ is

$$\alpha_\mu(\theta, r^\alpha) = \mu d\theta + \mu A_\alpha dr^\alpha.$$

- **Amended potential**: For $\mu \in \mathfrak{g}^*$, the amended potential is defined as

$$V_\mu(q) = V(q) + \frac{1}{2}\left\langle \mu, \mathbb{I}^{-1}(q)\mu\right\rangle = V(q) + \frac{1}{2}g_{00}^{-1}\mu^2.$$

- **Routhian**: The Routhian is a function on TQ defined as

$$R^{\mu} = \frac{1}{2}\|\mathrm{Hor}(q,\dot{q})\|^2 - V_{\mu},$$

where $\mathrm{Hor}(q,\dot{q}) = (-g_{00}^{-1}g_{0\alpha}\dot{r}^{\alpha}, \dot{r}^{\alpha})$ is the horizontal component of $(q,\dot{q})$ and the norm is given by the g_{ij} metric. Then, locally, we can write

$$R^{\mu} = \frac{1}{2}(g_{\alpha\beta} - g_{00}^{-1}g_{0\alpha}g_{0\beta})\dot{r}^{\alpha}\dot{r}^{\beta} - \frac{1}{2}g_{00}^{-1}\mu^2 - V(r^{\alpha}). \tag{13}$$

The general reduction theory[16,28] tells us that if a curve $q(t)$ in Q satisfying $\mathbf{J}_L(q,\dot{q}) = \mu$ is a solution of the Euler-Lagrange equations for the Lagrangian $L(q,\dot{q})$, then the induced curve on Q/G_{μ} satisfies the reduced Lagrangian variational principle; that is, the variational principle of Lagrange and d'Alembert on Q/G_{μ} with magnetic term $\mathcal{B}$ and the Routhian dropped to $T(Q/G_{\mu})$.

Let be $\hat{R}^{\mu}$ the reduced Routhian, that is the Routhian (13) dropped to the reduced space $\mathbf{J}_L^{-1}(\mu)/S^1$. Then (locally),

$$\hat{R}^{\mu} = \frac{1}{2}h_{\alpha\beta}\dot{r}^{\alpha}\dot{r}^{\beta} - V_{\mu}(r^{\alpha}), \tag{14}$$

where $h_{\alpha\beta} = g_{\alpha\beta} - g_{00}^{-1}g_{0\alpha}g_{0\beta}$ is a metric in the reduced space and $V_{\mu}(r^{\alpha})$ is the amended potential.

- **Equations of Lagrange-Routh**: The equations of motion in the reduced space $\mathbf{J}_L^{-1}(\mu)/S^1$ are given by

$$\frac{d}{dt}\frac{\partial \hat{R}^{\mu}}{\partial \dot{r}^{\alpha}} - \frac{\partial \hat{R}^{\mu}}{\partial r^{\alpha}} = -\mu B_{\alpha\beta}\dot{r}^{\beta},$$

where $B_{\alpha\beta} = \frac{\partial A_{\alpha}}{\partial r^{\beta}} - \frac{\partial A_{\beta}}{\partial r^{\alpha}}$ and $A_{\alpha} = g_{00}^{-1}g_{0\alpha}$.

More concretely,

$$h_{\alpha\beta}\ddot{r}^{\beta} + \left(\frac{\partial h_{\alpha\beta}}{\partial r^{\gamma}} - \frac{1}{2}\frac{\partial h_{\beta\gamma}}{\partial r^{\alpha}}\right)\dot{r}^{\beta}\dot{r}^{\gamma} = -\frac{\partial V_{\mu}(r^{\alpha})}{\partial r^{\alpha}} - \mu B_{\alpha\beta}\dot{r}^{\beta}. \tag{15}$$

B. Cotangent Bundle Reduction

Now, we perform the corresponding reduction in the Hamiltonian side. See Ref. 15 for the details.

Let us consider that the initial Hamiltonian can be written (locally) as

$$H(r^\alpha, p_\theta, p_r^\alpha) = \frac{1}{2} g^{00} p_\theta^2 + g^{0\alpha} p_\theta p_\alpha + \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V(r^\alpha),$$

where the metric elements g^{ij} correspond to the inverse of the metric g_{ij}. That is, $g_{ij} g^{jk} = \delta_i^k$, where $i, j, k = 0, \ldots, n$, and δ_i^k denotes the Kronecker delta function.

Thus, we assume that the initial Hamiltonian is invariant under the action of the abelian symmetry group $G = SO(2) = S^1$.

Let us perform the computations of all the extra ingredients needed in the reduction in the Hamiltonian side[15] :

- **Momentum map**: The momentum map corresponds to the angular momentum of the system: $\mathbf{J} : T^*Q \longrightarrow \mathfrak{g}^*$

  $$\langle \mathbf{J}(\theta, r^\alpha, p_\theta, p_{r^\alpha}), \xi \rangle = \langle (p_\theta, p_\alpha), (\xi, 0) \rangle = p_\theta \xi.$$

  Thus, $\mathbf{J}(\theta, r^\alpha, p_\theta, p_{r^\alpha}) = p_\theta$.

- **Momentum shifting**: In this case, it is convenient to perform a shift of the momenta from $\mathbf{J}^{-1}(\mu)$ to $\mathbf{J}^{-1}(0)$ (and also in the corresponding reduced spaces) in the following way:

  $$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) = \{(\theta, r^\alpha, \mu, p_\alpha)\} & \xrightarrow{t^\mu} & \mathbf{J}^{-1}(0) = \{(\theta, r^\alpha, 0, \tilde{p}_\alpha)\} \\ \downarrow & & \downarrow \\ \mathbf{J}^{-1}(\mu)/G_\mu = \mathbf{J}^{-1}(\mu)/S^1 & \xrightarrow{t^\mu_G} & \mathbf{J}^{-1}(0)/S^1 = \mathbf{J}^{-1}(0)/G \end{array}$$

  where

  $$\begin{aligned} t^\mu(\theta, r^\alpha, \mu, p_\alpha) &= (\theta, r^\alpha, \mu, p_\alpha) - (\theta, r^\alpha, \mu, \mu A_\alpha) \\ &= (\theta, r^\alpha, 0, p_\alpha - \mu A_\alpha) = (\theta, r^\alpha, \tilde{p}_\theta, \tilde{p}_\alpha). \end{aligned}$$

  Thus, the shifting is given by $\tilde{p}_\alpha = p_\alpha - \mu A_\alpha$ and $\tilde{p}_\theta = 0$.

- **Reduced Hamiltonian**: In $\mathbf{J}^{-1}(0)/G$, we have $H_{\alpha_\mu} = \frac{1}{2}\|\tilde{p}\|^2 + V_\mu$, where $\|\cdot\|$ is the norm related to the g^{ij} metric and $V_\mu = V + \frac{1}{2}\mathbb{I}^{-1}\mu^2$ is the amended potential. Thus, recalling that $\tilde{p}_\theta = 0$, the Hamiltonian in the reduced space $\mathbf{J}^{-1}(0)/G$ is:

$$H_\mu(r^\alpha, p_\alpha) = \frac{1}{2}g^{\alpha\beta}\tilde{p}_\alpha\tilde{p}_\beta + V(r^\alpha) + \frac{1}{2}\mu^2 g_{00}^{-1}.$$

- **Reduced Symplectic Form**: In general, in the reduced space, the symplectic form is not canonical. The projection is given by the map:

$$((T^*Q)_\mu, \Omega_\mu) \xrightarrow{P_\mu} ((T^*(Q/G), \omega - B_\mu)$$

where the "reduced" symplectic form is

$$\omega_\mu = \omega - B_\mu = dr^\alpha \wedge d\tilde{p}_\alpha - \mu\frac{\partial A_\alpha}{\partial r^\beta}dr^\beta \wedge dr^\alpha.$$

- **Hamiltonian equations**: The Hamiltonian equations are given by[16]

$$\mathbf{i}_{(\dot{r}^\alpha\partial_{r^\alpha} + \dot{\tilde{p}}_\alpha\partial_{\tilde{p}_\alpha})}\omega_\mu = dH_\mu,$$

where $\mathbf{i}_X\Omega$ denotes the interior product (or contraction) of the vector field X and the 1-form Ω. Computing both sides of this identity,

$$\begin{aligned}
\mathbf{i}_{(\dot{r}^\alpha\partial_{r^\alpha} + \dot{\tilde{p}}_\alpha\partial_{\tilde{p}_\alpha})}\omega_\mu &= \dot{r}^\alpha d\tilde{p}_\alpha - \mu\frac{\partial A_\alpha}{\partial r^\beta}\dot{r}^\beta dr^\alpha + \mu\frac{\partial A_\alpha}{\partial r^\beta}\dot{r}^\alpha dr^\beta - \dot{\tilde{p}}_\alpha dr^\alpha, \\
dH_\mu &= \frac{\partial H_\mu}{\partial r^\alpha}dr^\alpha + \frac{\partial H_\mu}{\partial \tilde{p}_\alpha}d\tilde{p}_\alpha,
\end{aligned}$$

we obtain the equations of motion in the reduced $J^{-1}(0)/G$ space:

$$\dot{r}^\alpha = \frac{\partial H_\mu}{\partial \tilde{p}_\alpha}, \qquad \dot{\tilde{p}}_\alpha = -\frac{\partial H_\mu}{\partial r^\alpha} - \mu\left(\frac{\partial A_\alpha}{\partial r^\beta} - \frac{\partial A_\alpha}{\partial r^\beta}\right)\dot{r}^\beta.$$

Finally, we can write them more explicitly as:

$$\dot{r}^\alpha = g^{\alpha\beta}\tilde{p}_\beta, \qquad \dot{\tilde{p}}_\alpha = -\frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial r^\alpha}\tilde{p}_\beta\tilde{p}_\gamma - \frac{\partial V(r^\alpha)}{\partial r^\alpha} - \frac{1}{2}\mu^2\frac{\partial g_{00}^{-1}}{\partial r^\alpha} - \mu B_{\alpha\beta}g^{\beta\gamma}\tilde{p}_\gamma. \tag{16}$$

C. Reduced Legendre transformation

Finally, the correspondence between the reduced equations of motion on the Hamiltonian and Lagrangian sides is given by the reduced Legendre transform. We start with the reduced Routhian (14),

$$\hat{R}^\mu = \frac{1}{2}(g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta})\dot{r}^\alpha \dot{r}^\beta - \frac{1}{2} g_{00}^{-1} \mu^2 - V(r^\alpha)$$

and the shifted momenta,

$$\tilde{p}_\alpha = \frac{\partial \hat{R}^\mu}{\partial \dot{r}^\alpha} = (g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta})\dot{r}^\beta.$$

Using the identities

$$\begin{aligned} g_{0\alpha} g^{0\beta} + g_{\alpha\gamma} g^{\gamma\beta} &= \delta_\alpha^{\ \beta}, \\ g^{\alpha\beta} g_{\beta 0} + g^{\alpha 0} g_{00} &= 0, \end{aligned} \tag{17}$$

we obtain the first equation in (16): $g^{\alpha\beta}\tilde{p}_\beta = \dot{r}^\alpha$.

Now, in order to recover the reduced Lagrange-Routh equations (15), we compute the time-derivative of the shifted momenta

$$\dot{\tilde{p}}_\alpha = (g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta})\ddot{r}^\beta + \frac{\partial}{\partial r^\gamma}\left(g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta}\right)\dot{r}^\gamma \dot{r}^\beta,$$

and the derivative with respect to r^α of the identities (17),

$$\begin{aligned} \frac{\partial g_{0\epsilon}}{\partial r^\alpha} g^{0\beta} + g_{0\epsilon}\frac{\partial g^{0\beta}}{\partial r^\alpha} + \frac{\partial g_{\epsilon\gamma}}{\partial r^\alpha} g^{\gamma\beta} + g_{\epsilon\gamma}\frac{\partial g^{\gamma\beta}}{\partial r^\alpha} &= 0, \\ \frac{\partial g^{\beta\gamma}}{\partial r^\alpha} g_{\gamma 0} + g^{\beta\gamma}\frac{\partial g_{\gamma 0}}{\partial r^\alpha} + \frac{\partial g^{\beta 0}}{\partial r^\alpha} g_{00} + g^{\beta 0}\frac{\partial g_{00}}{\partial r^\alpha} &= 0. \end{aligned}$$

If we substitute the last three identities together with (17) in the second equation of (16), we obtain:

$$\begin{aligned} (g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta})\ddot{r}^\beta + \frac{\partial}{\partial r^\gamma}\left(g_{\alpha\beta} - g_{00}^{-1} g_{0\alpha} g_{0\beta}\right)\dot{r}^\gamma \dot{r}^\beta &= \\ \frac{1}{2}\frac{\partial}{\partial r^\alpha}\left(g_{\beta\gamma} - g_{00}^{-1} g_{0\beta} g_{0\gamma}\right)\dot{r}^\beta \dot{r}^\gamma - \frac{\partial V(r^\alpha)}{\partial r^\alpha} - \frac{1}{2}\mu^2 \frac{\partial g_{00}^{-1}}{\partial r^\alpha} - \mu B_{\alpha\beta}\dot{r}^\beta, & \end{aligned}$$

which exactly corresponds to the Lagrange-Routh equations (15).

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