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Stable manifolds associated to fixed points with linear part equal to identity

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Abstract

We consider maps defined on an open set of $\mathbb{R}^{n+m}$ having a fixed point whose linear part is the identity. We provide sufficient conditions for the existence of a stable manifold in terms of the nonlinear part of the map.

These maps arise naturally in some problems of Celestial Mechanics. We apply the results to prove the existence of parabolic orbits of the spatial elliptic three-body problem.
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1. Introduction

It is well known that invariant manifolds associated to invariant objects (fixed points, periodic orbits, etc.) of a dynamical system yield essential information for the analysis of the dynamical structure of the system. When an invariant object satisfies some kind of hyperbolicity there are many results concerning the existence, regularity and uniqueness of their invariant manifolds, see for instance [6,8–10].

The case of invariant objects without hyperbolic "directions" is more complicated. The full neighborhood of the object is a central manifold. If we consider dynamical systems generated by maps, the fact that a neighborhood of the fixed point is a central manifold means that all the eigenvalues of the linear part of the map at the fixed point have modulus one.

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The case that all eigenvalues are exactly equal to one is the most degenerate one. In this case the set of points whose positive iterates converge to the fixed point may have nonempty interior. This set is invariant by the map. We can call it stable invariant set or stable invariant manifold in some generalized sense. In the analogous way we can define the unstable invariant set.

The problem of deciding whether a parabolic fixed point of a vector field or a map has associated stable and unstable manifolds (inside the central manifold), has not been solved in general, but there are already some existence and uniqueness results for these manifolds. For two-dimensional maps with fixed points with identity linear part we mention [13,16]. For two-dimensional maps with linear part equal to:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

we refer to [2,7]. In this context, it may happen that both the stable and the unstable invariant sets are open sets. See an example of such case in [7]. Some stable manifolds theorems for a class of systems coming from problems in Celestial Mechanics can be found in [4,15]. In all these problems the stable manifolds are one dimensional.

Maps having parabolic fixed points appear in applied problems. For example, when studying parabolic and oscillatory orbits in some problems of Celestial Mechanics. The most studied case has been the planar three-body problem. In the planar three-body problem a parabolic orbit is a trajectory of a particle arriving to infinity with zero speed, while the trajectories of the other two particles remain bounded for all positive times. An orbit of the planar three-body problem is called oscillatory if the upper limit (along time) of the distance between particles is infinite, but the lower limit is finite. Thus it seems clear that the oscillatory orbits come and go infinitely often going (somehow) to infinity. Hence a good way to look at this problem is to look for solutions that are "homoclinic at infinity". Therefore it seems natural to associate to the infinity some invariant object, through the introduction of a special set of coordinates. This object is usually called the infinity manifold. In the case of the planar three-body problem McGehee and Easton [5] prove that the infinity set may be seen as a three-sphere foliated by periodic orbits. McGehee [13] considers three problems: the restricted three-body problem, the Sitnikov problem and the one-dimensional three-body problem, and proves, after certain changes of variables, that infinity may be reduced to a periodic orbit. Later, Martínez and Pinyol [12] prove, among other things, that in the elliptic restricted three-body problem the infinity manifold is also foliated by periodic orbits. Using the existence theorem of invariant manifolds given in [13], Delgado and Vidal [3], also prove the existence of parabolic orbits in the tetrahedral four-body problem and, finally, Álvarez and Llibre [1], consider the same question for the elliptic collision restricted three-body problem, which consists in two bodies of equal masses in a collision elliptic orbit, while their centre of mass is at rest and a third particle of zero mass moving in a perpendicular line to the line of motion of the other two.

An approach to the search of oscillatory orbits is to prove that these periodic orbits, which represent infinity in the original system, have transversely intersecting stable and unstable manifolds. This is not a sufficient condition (see [5]), but it seems to be necessary to prove the existence of oscillatory orbits. In the problems treated by McGehee in [13] and in the elliptic restricted problem in [12] the existence of these homoclinic solutions implies the existence of oscillatory orbits.

In all these examples, the periodic orbits in the infinity manifold are degenerate in the sense that the derivative of the Poincaré map associated to them has an eigenvalue equal to one. Using the existence theorems proved in [4,13,15] it is possible to prove that parabolic orbits form a smooth manifold. Robinson [15], Xia [17], Martínez and Pinyol [12] and Moeckel [14] prove the existence of heteroclinic orbits and consequently they can conclude that there exist oscillatory orbits in some instances of the three-body problem.

Here we generalize results on existence and analyticity of invariant manifolds of several papers starting with [13], from two-dimensional to $(n+m)$-dimensional maps.

We consider maps in $\mathbb{R}^n \times \mathbb{R}^m$ with the origin fixed and its linear part equal to the identity. Under suitable conditions on the nonlinear terms we establish the existence of n-dimensional stable manifolds expressed as graphs of functions defined in domains which have the fixed point on its boundary. In Section 3 we deal with the Lipschitz case and in Section 4 we consider the analytic case.

The methods we use in this work are generalizations of the ones of McGehee [13] but we need to introduce extra arguments based on degree theory that in one-dimensional invariant manifolds reduce to elementary observations.

Section 5 contains the examples. The first one is just a simple application, and the second one consists in looking for parabolic orbits in the spatial three-body problem. In this problem, we prove that the parabolic orbits form an analytic manifold of dimension two in the phase space. For this reason the known existence theorems do not apply in this case.

2. Definitions and notation

We consider maps $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ of the form

$$F(x,y) = (x + p(x,y) + f(x,y), y + q(x,y) + g(x,y)), \tag{2.1}$$

where $p(x,y), q(x,y)$ are homogeneous polynomials of degree N_p, N_q respectively with N_p, $N_q \geqslant 2$, $f(x,y)$, $g(x,y)$ are differentiable functions of orders $o(||(x,y)||^{N_p})$ and $o(||(x,y)||^{N_q})$ and their derivatives $Df(x,y)$, $Dg(x,y)$ are $o(||(x,y)||^{N_p-1})$ and $o(||(x,y)||^{N_q-1})$ respectively.

We introduce the projectors: $\pi^1(x,y) = x$, and $\pi^2(x,y) = y$. Given a subset $V \subset \mathbb{R}^n$ we define

$$W^s_V = \{(x,y) \in U : \pi^1 F^k(x,y) \in V,\ \ k \geqslant 0,\ \ F^k(x,y) \to 0,\ \text{as } k \to \infty\} \tag{2.2}$$

and its local version

$$W^s_{V,r} = \{(x,y) \in U : \pi^1 F^k(x,y) \in V \cap B(0,r),\ k \geqslant 0,\ F^k(x,y) \to 0,\ \text{as}\ k \to \infty\}. \quad (2.3)$$

These definitions depend on the decomposition $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. Particular cases are $n = 1$ or $m = 1$. In the two-dimensional case, if $n = 1$ and $m = 1$, V can be taken as the intervals $(0, r)$ or $(-r, 0)$. When $V = (0, r)$ the corresponding invariant manifold is denoted by W^{s+} in [7].

Let $||\cdot||$ be a norm in $\mathbb{R}^k$. Given $V \subset \mathbb{R}^n$, we introduce the following notation:

$$V(r) = \{x \in V : ||x|| < r\}, \quad V^1(\rho) = \overline{\{\rho x/||x|| : x \in V(r)\}}.$$

Notice that if $r_1 < r_2$ then $V(r_1) \subset V(r_2)$.

Also we introduce the following sets:

$$V(r,\beta) = \{(x,y) \in \mathbb{R}^{n+m} : x \in \overline{V(r)}, ||y|| \leqslant \beta ||x||\},$$

$$V^+(r,\beta) = \{(x,y) \in \mathbb{R}^{n+m} : x \in \overline{V(r)}, ||y|| \geqslant \beta ||x||\},$$

$$v^+(r,\beta) = \{(x,y) \in \mathbb{R}^{n+m} : (x,y) \in V(r,\beta), ||y|| = \beta ||x||\},$$

$$S(\alpha) = \{(\xi,\eta) \in \mathbb{R}^{n+m} : ||\eta|| \geqslant \alpha ||\xi||\}.$$

In order to illustrate the previous definitions, we provide Figs. 1 and 2.

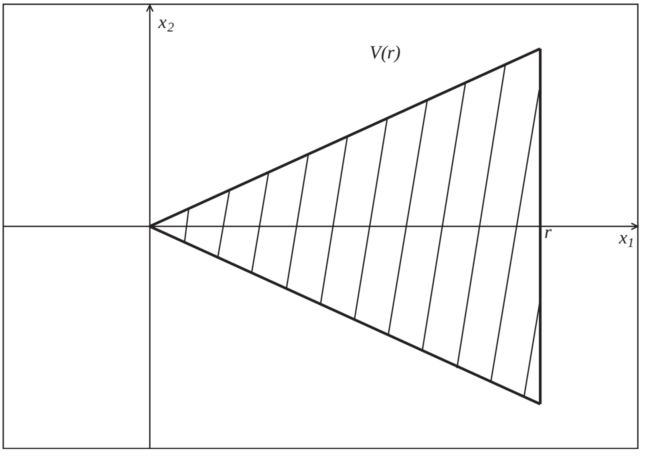

Fig. 1. Example of a set $V(r)$ in $\mathbb{R}^2$.

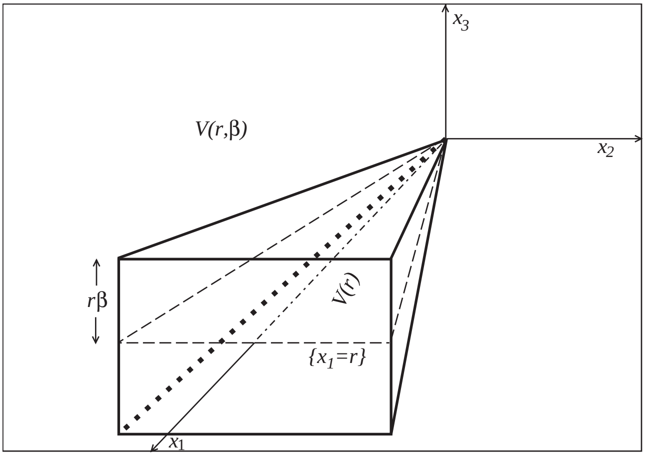

Fig. 2. A set $V(r,\beta)$ in $\mathbb{R}^3$, with the supremum norm.

3. The Lipschitz case

This section is devoted to prove, under suitable hypotheses, the existence of a Lipschitz stable invariant manifold in the sense of definitions (2.2) and (2.3) for maps F of the form (2.1).

We will assume that there exists a set $V \subset U$ and $r, \rho > 0$ such that:

H1 The polynomial p satisfies $\sup_{x\in V^1(\rho)} \|\mathrm{Id} + D_x p(x,0)\| < 1$.
H2 The polynomial q satisfies $D_x q(x,0) = 0$ for $x \in V^1(\rho)$ and $\sup_{x\in V^1(\rho)} \|\mathrm{Id} - D_y q(x,0)\| < 1$.
H3 There exists $A > 0$ such that for all $x \in V(r)$, $\mathrm{dist}(x + p(x,0), V(r)^c) \geqslant A\|x\|^{N_p}$.

Note that H2 implies that $q(x,0) = 0$. The main theorem of this section is:

Theorem 3.1. *Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ be a map of class C^1, of the form*

$$(x,y) \mapsto (x + p(x,y) + f(x,y), y + q(x,y) + g(x,y)) \tag{3.1}$$

where $p(x,y), q(x,y)$ are homogeneous polynomials of degree N_p and N_q respectively ($N_p, N_q \geqslant 2$), $f(x,y)$ is of order $o(\|(x,y)\|^{N_p})$, $Df(x,y)$ is of order $o(\|(x,y)\|^{N_p-1})$, $g(x,y)$ is of order $o(\|(x,y)\|^{N_q})$ and $Dg(x,y)$ is of order $o(\|(x,y)\|^{N_q-1})$.

Then, if there exists a convex open set $V \subset \mathbb{R}^n$, $0 \in \partial V$ and $r, \rho > 0$ such that hypotheses H1–H3 *hold, $W^s_{V,r}$ is the graph of a Lipschitz function*

$$\varphi : V(r) \to \mathbb{R}^m.$$

Remark 3.2. Hypotheses H1 and H2 provide, through the nonlinear terms, a kind of weak hyperbolicity for the fixed points in a suitable domain.

Remark 3.3. An unstable manifold theorem can be obtained immediately by considering the inverse map.

Remark 3.4. Since $V^1(\rho)$ is compact, there exists a positive constant M such that

$$||\mathrm{Id} + D_x p(x,0)|| - 1 < -M, \quad ||\mathrm{Id} - D_y q(x,0)|| - 1 < -M$$

for all $x \in V^1(\rho)$. This implies that if $r \leqslant 1$ and $t \in [0,1]$

$$||\mathrm{Id} + tD_x p(x,0)|| - 1 < -tM||x||^{N_p-1}, \quad ||\mathrm{Id} - tD_y q(x,0)|| - 1 < -tM||x||^{N_q-1}$$

for all $x \in V(r)$. Indeed, the inequality $||\lambda\,\mathrm{Id} + D_x p(x,0)|| - ||\mu\,\mathrm{Id} + D_x p(x,0)|| \leqslant |\lambda - \mu|$ implies that $||\lambda\,\mathrm{Id} + D_x p(x,0)|| - \lambda$ is a decreasing function of λ. Then, if $x \in V(r)$ and $t \in (0,1]$,

$$\begin{aligned} ||\mathrm{Id} + tD_x p(x,0)|| - 1 &= \left|\left|\mathrm{Id} + t||x||^{N_p-1} D_x p\left(\frac{x}{||x||}, 0\right)\right|\right| - 1 \\ &= t||x||^{N_p-1}\left[\left|\left|\frac{1}{t||x||^{N_p-1}}\,\mathrm{Id} + D_x p\left(\frac{x}{||x||}, 0\right)\right|\right| - \frac{1}{t||x||^{N_p-1}}\right] \\ &\leqslant -t||x||^{N_p-1} M. \end{aligned} \tag{3.2}$$

The second inequality follows in the same way.

The rest of this section is devoted to prove Theorem 3.1. The main idea of the proof consists in, given $x_0 \in V(r)$, looking for the set of points of the form (x_0, y) whose all positive iterates remain in a neighborhood of the origin and converge to it.

We claim that this set reduces to a unique point (x_0, y_0). Hence there exists a function $y_0 = \varphi(x_0)$ whose graph is the stable manifold.

To prove the claim, as well as the fact that φ is Lipschitz, we will consider a sequence of nested sets defined as the sets of points whose first k iterates remain in a neighborhood of the origin.

To control this sequence we need a series of preparatory lemmas which provide us with some quantitative estimates of the weak hyperbolicity generated by the nonlinear terms of the map outside the origin.

Lemma 3.6 provides bounds for the contraction and expansion of the linearized map along the x- and y-axis, respectively. Lemmas 3.7 and 3.8 extend these estimates from the linearization to the map itself. Lemmas 3.9 and 3.10 study how the derivative acts on vectors of the tangent space, in particular Lemma 3.9 establishes that there exists and invariant cone for DF. Lemma 3.13 will be applied as an iterative lemma to control the differences of the iterates of two initial points. The nested sets of the sequence are constructed iteratively. It is essential that they do not become void at some level of the process. This is guaranteed by Lemma 3.15.

In all next lemmas we will assume implicitly the hypotheses of Theorem 3.1.

Lemma 3.5. *If $r>0$ and $\beta>0$ are small enough, then we have that*

$$\pi^1 F(x,y) \in V(r), \quad \text{for } (x,y) \in V(r,\beta).$$

Proof. It is a consequence of hypothesis H3. Note that for all $(x,y) \in V(r,\beta)$

$$\begin{aligned} ||\pi^1 F(x,y) - x - p(x,0)|| &\leqslant ||p(x,y) - p(x,0)|| + ||f(x,y)|| \\ &\leqslant \sup_{||\xi|| \leqslant ||y||} ||D_y p(x,\xi)|| \, ||y|| + \eta ||x||^{N_p} \leqslant (C\beta + \eta)||x||^{N_p} \end{aligned}$$

with suitable C and arbitrarily small η, if r and β are small enough. Hence

$$\begin{aligned} \operatorname{dist}(\pi^1 F(x,y), V(r)^c) &\geqslant \operatorname{dist}(x + p(x,0), V(r)^c) - ||\pi^1 F(x,y) - x - p(x,0)|| \\ &\geqslant A||x||^{N_p} - (C\beta + \eta)||x||^{N_p} > 0, \end{aligned}$$

if $(C\beta + \eta) < A$ which implies that $\pi^1 F(x,y) \in V(r)$. $\square$

Lemma 3.6. *There exist constants K_1 and K_2 such that for $(x,y) \in V(r,\beta)$ and for $t \in [0,1]$,*

(1) $||\mathrm{Id} + tD_x p(x,y) + tD_x f(x,y)|| \leqslant 1 - K_1 t ||x||^{N_p - 1}$,
(2) $||(\mathrm{Id} + tD_y q(x,y) + tD_y g(x,y))^{-1}|| \leqslant 1 - K_2 t ||x||^{N_q - 1}$.

Proof. (1) Since p is homogeneous there exists $K>0$ such that $||D^2 p(x,y)|| \leqslant K||x||^{N_p-2}$. By the conditions over f, given $\eta>0$ there exists $r>0$ such that $||D_x f(x,y)|| \leqslant \eta ||x||^{N_p-1}$ for $(x,y) \in V(r,\beta)$. Then, using Remark 3.4,

$$\begin{aligned} ||\mathrm{Id} + tD_x p(x,y) + tD_x f(x,y)|| &\leqslant ||\mathrm{Id} + tD_x p(x,0)|| + t||D_x p(x,y) - D_x p(x,0)|| \\ &\quad + t||D_x f(x,y)|| \\ &\leqslant 1 - tM||x||^{N_p-1} + tK\beta||x||^{N_p-1} + t\eta||x||^{N_p-1} \\ &\leqslant 1 - tK_1||x||^{N_p-1} \end{aligned}$$

with $K_1>0$, if we take β and η small enough.

(2) In the same way as in (1) we can prove that

$$||\mathrm{Id} - tD_y q(x,y) - tD_y g(x,y)|| \leqslant 1 - tK_0||x||^{N_q-1} \tag{3.3}$$

if we take β and r small enough. The result follows because there exists K_0' such that

$$||(\mathrm{Id}+tD_yq(x,y)+tD_yg(x,y))^{-1}-(\mathrm{Id}-tD_yq(x,y)-tD_yg(x,y))||$$

$$\leqslant t^2K_0'||x||^{2N_q-2}. \qquad \square$$

Lemma 3.7. *There exists a constant M_1 such that for $(x,y)\in V(r,\beta)$ and for any $t\in[0,1]$,*

$$||x+tp(x,y)+tf(x,y)||\leqslant||x||(1-tM_1||x||^{N_p-1}).$$

In particular, for $t=1$ we have that $||\pi^1F(x,y)||<||x||$.

Proof. By the mean value theorem and (3.2) we have that

$$||x+tp(x,0)||\leqslant\int_0^1||\mathrm{Id}+tD_xp(sx,0)||\,||x||\,ds\leqslant\int_0^1(1-tM||sx||^{N_p-1})||x||\,ds$$

$$=\left(1-tM\frac{1}{N_p}||x||^{N_p-1}\right)||x||.$$

Let $M_0=M/N_p$. Moreover, there exists $K>0$ such that $||D_yp(x,y)||\leqslant K||x||^{N_p-1}$ if $(x,y)\in V(r,\beta)$ and, given $\eta>0$, there exists $r>0$ such that $||f(x,y)||\leqslant\eta||x||^{N_p}$ for $(x,y)\in V(r,\beta)$. Then

$$||x+tp(x,y)+tf(x,y)||\leqslant||x+tp(x,0)||+t||p(x,y)-p(x,0)||+t||f(x,y)||$$

$$\leqslant(1-tM_0||x||^{N_p-1})||x||+t\beta K||x||^{N_p}+t\eta||x||^{N_p}$$

$$\leqslant(1-tM_1||x||^{N_p-1})||x||$$

with $M_1>0$ if we take β and η small enough. $\square$

Lemma 3.8. *There exists $M_2>0$ such that for any $(x,y)\in v^+(r,\beta)$ and for any $0\leqslant t\leqslant1$ we have*

$$||y+tq(x,y)+tg(x,y)||\geqslant||y||(1+tM_2||x||^{N_q-1}).$$

Proof. Since $q(x,0)=0$, if we call $\phi(y)=y+tq(x,y)$ then $\phi(0)=0$ and $y=\phi^{-1}\circ\phi(y)=\int_0^1D\phi^{-1}(s\phi(y))\phi(y)\,ds$. This implies

$$||\phi(y)||\geqslant\left(\int_0^1||D\phi^{-1}(s\phi(y))||\,ds\right)^{-1}||y||$$

and since $D\phi^{-1}(s\phi(y)) = (D\phi(\phi^{-1}(s\phi(y))))^{-1}$ and $\phi^{-1}(s\phi(y)) \in V(r,\beta')$ with $\beta' \geqslant \beta$, applying (2) of Lemma 3.6 with $g=0$ we get $||\phi(y)|| \geqslant (1 - tK_2||x||^{N_q-1})^{-1}||y||$.

On the other hand, for all $\eta > 0$, there exists $r > 0$ such that $||g(x,y)|| \leqslant \beta\eta||x||^{N_q}$ if $(x,y) \in V(r,\beta)$. Assume that $||y|| = \beta||x||$. Then $||g(x,y)|| \leqslant \beta\eta||x||^{N_q} = \eta||y||\ ||x||^{N_q-1}$. Therefore $||\phi(y) + tg(x,y)|| \geqslant (1 - tK_2||x||^{N_q-1})^{-1}||y|| - \eta||y||\ ||x||^{N_q-1} \geqslant ||y||(1 + tM_2||x||^{N_q-1})$ with $M_2 > 0$ if we take η and r small enough. $\square$

Lemma 3.9. *There exist $r > 0$, $\beta > 0$ and $\alpha \in (0,1]$ such that for all $(x,y) \in V(r,\beta)$, $DF(x,y) : S(\alpha) \to S(\alpha)$. In fact we have that, for $\zeta \in S(\alpha)$*

$$\alpha||\pi^1 DF(x,y)\zeta|| \leqslant ||\pi^2\zeta|| \quad \textit{and} \quad ||\pi^2 DF(x,y)\zeta|| \geqslant ||\pi^2\zeta||. \tag{3.4}$$

Proof. Let $\zeta = (\xi,\eta) \in S(\alpha)$ and let $K > 0$ be such that $||D_y p(x,y) + D_y f(x,y)|| \leqslant K||x||^{N_p-1}$ for $(x,y) \in V(r,\beta)$. Using (1) of Lemma 3.6 it is clear that

$$\begin{aligned}\alpha||\pi^1 DF(x,y)\zeta|| &= \alpha||(\mathrm{Id} + D_x p + D_x f)\xi + (D_y p + D_y f)\eta|| \\ &\leqslant (1 - K_1||x||^{N_p-1})||\eta|| + \alpha K||x||^{N_p-1}||\eta|| \\ &\leqslant ||\eta||\end{aligned}$$

if we take $\alpha \leqslant K_1/K$.

The second inequality in (3.4) is proved in the same way, using (2) of Lemma 3.6 and that, since $D_x q(x,0) = 0$ and $D_x g(x,y) = o(||(x,y)||^{N_q-1})$, there exists $K > 0$ such that

$$||D_x q(x,y) + D_x g(x,y)|| \leqslant K\beta||x||^{N_q-1} \tag{3.5}$$

if r is small enough. $\square$

Lemma 3.10. *If $r,\beta > 0$ are small enough, $(x,y) \in V(r,\beta)$ and $\zeta \in S(\alpha)$ we have that*

$$||\pi^2 DF^{-1}(x,y)\zeta|| \leqslant ||\pi^2\zeta||.$$

Proof. It is clear that F is locally invertible in a neighborhood of the origin and that F^{-1} is defined in a set of the form $V(r,\beta)$. Moreover F^{-1} can be written as

$$F^{-1}(x,y) = (x - p(x,y) + \tilde{f}(x,y), y - q(x,y) + \tilde{g}(x,y))$$

with $\tilde{f}(x,y) = o(||(x,y)||^{N_p})$, $\tilde{g}(x,y) = o(||(x,y)||^{N_q})$, $D\tilde{f}(x,y) = o(||(x,y)||^{N_p-1})$ and $D\tilde{g}(x,y) = o(||(x,y)||^{N_q-1})$. Let $\zeta = (\xi,\eta) \in S(\alpha)$. Then, using (3.3) and (3.5) there exist β and r small enough such that

$$\begin{aligned}||\pi^2 DF^{-1}(x,y)\zeta|| &\leqslant ||(\mathrm{Id} - D_y q + D_y\tilde{g})\eta|| + ||(-D_x q + D_x\tilde{g})\xi|| \\ &\leqslant (1 - K_0||x||^{N_q-1})||\eta|| + K\frac{\beta}{\alpha}||x||^{N_p-1}||\eta|| \\ &\leqslant ||\eta||. \qquad \square\end{aligned}$$

Lemma 3.11. *Given* $l \in \mathbb{N}$, $z_k \in V(r,\beta)$ *for all* $k \in \{1,\dots,l\}$ *and* $\zeta \in S(\alpha)$, *we have that* $(1/l)\sum_{k=1}^{l} DF(z_k)\zeta \in S(\alpha)$.

Proof. Let $\zeta = (\xi,\eta) \in S(\alpha)$. Applying estimate (3.4) of Lemma 3.9, we obtain that $\alpha||(1/l)\sum_{k=1}^{l}\pi^1 DF(z_k)\zeta|| \leqslant (1/l)\sum_{k=1}^{l}\alpha||\pi^1 DF(z_k)\zeta|| \leqslant ||\eta||$. On the other hand, if we denote $Q_l = \frac{1}{l}\sum_{k=1}^{l}(D_y q(z_k) + D_y g(z_k))$, by (3.3)

$$||\mathrm{Id} - Q_l|| \leqslant (1/l)\sum_{k=1}^{l} ||\mathrm{Id} - (D_y q(z_k) + D_y g(z_k))|| \leqslant 1 - (K_0/l)\sum_{k=1}^{l} ||x_k||^{N_q-1}.$$

Therefore, $||(\mathrm{Id} + Q_l)^{-1}|| \leqslant 1 - \frac{1}{l}\frac{K_0}{2}\sum_{k=1}^{l}||x_k||^{N_q-1}$, which implies that

$$||(\mathrm{Id} + Q_l)\eta|| \geqslant (1 + K_0/(2l))\sum_{k=1}^{l} ||x_k||^{N_q-1})||\eta|| \tag{3.6}$$

if we take β and r small enough. Then, from (3.6) and (3.5), we obtain

$$\begin{aligned}&\left|\left|\frac{1}{l}\sum_{k=1}^{l}\pi^2 DF(z_k)\zeta\right|\right| \\ &\geqslant ||(\mathrm{Id} + Q_l)\eta|| - \left|\left|\left(\frac{1}{l}\sum_{k=1}^{l}(D_x q(z_k) + D_x g(z_k))\right)\xi\right|\right| \\ &\geqslant \left(1 + \frac{1}{l}\frac{K_0}{2}\sum_{k=1}^{l}||x_k||^{N_q-1}\right)||\eta|| - \frac{K}{l}\frac{\beta}{\alpha}\sum_{k=1}^{l}||x_k||^{N_q-1}||\eta|| \\ &\geqslant \left(1 + \frac{1}{l}\left[\frac{K_0}{2} - K\frac{\beta}{\alpha}\right]\sum_{k=1}^{l}||x_k||^{N_p-1}\right)||\eta|| \\ &\geqslant ||\eta||\end{aligned}$$

if we take β small enough. $\square$

Lemma 3.12. *Let r and β be small enough. Let $z^1, z^2 \in V(r,\beta)$ be two different points such that $z^2 - z^1 \in S(\alpha)$. Then, there exists a constant $c \geqslant 1$ such that if $\alpha > c\beta$*

$$z^1 + t(z^2 - z^1) \in V(r, c\beta) \quad \textit{for all } t \in [0,1].$$

Proof. Let c_1 and c_2 be such that $c_1 ||u||_2 \leqslant ||u|| \leqslant c_2 ||u||_2$, for $u \in \mathbb{R}^k$, where $||\cdot||_2$ denotes the euclidean norm. We take $c = c_2/c_1$ and we put $z^i = (x^i, y^i)$ for $i = 1, 2$. Since, by hypothesis, V is convex, it remains to see that

$$||y^1 + t(y^2 - y^1)|| \leqslant c\beta ||x^1 + t(x^2 - x^1)||, \quad \forall t \in [0,1].$$

We note that, if $\alpha c_1 \geqslant \beta c_2$,

$$\begin{aligned} ||y^1 - y^2||_2 &\geqslant \frac{1}{c_2} ||y^1 - y^2|| \geqslant \alpha \frac{1}{c_2} ||x^1 - x^2|| \\ &\geqslant \alpha \frac{c_1}{c_2} ||x^1 - x^2||_2 \geqslant \beta ||x^1 - x^2||_2. \end{aligned} \tag{3.7}$$

By (3.7)

$$\begin{aligned} \beta^2 \langle x^1, x^2 \rangle - \langle y^1, y^2 \rangle &= \frac{1}{2} [\beta^2 (||x^1||_2^2 + ||x^2||_2^2 - ||x^1 - x^2||_2^2) \\ &\quad - (||y^1||_2^2 + ||y^2||_2^2 - ||y^1 - y^2||_2^2)] \\ &\geqslant \frac{1}{2} [||y^1 - y^2||_2^2 - \beta^2 ||x^1 - x^2||_2^2] \\ &\geqslant 0. \end{aligned} \tag{3.8}$$

Then, using (3.8), we have that for z^1, $z^2 \in V(r,\beta)$,

$$\begin{aligned} &\beta^2 ||x^1 + t(x^2 - x^1)||_2^2 - ||y^1 + t(y^2 - y^1)||_2^2 \\ &= t^2 (\beta^2 ||x^2||_2^2 - ||y^2||_2^2) + (1-t)^2 (\beta^2 ||x^1||_2^2 - ||y^1||_2^2) \\ &\quad + 2t(1-t)(\beta^2 \langle x^1, x^2 \rangle - \langle y^1, y^2 \rangle) \geqslant 0. \end{aligned}$$

Translating this condition to the original norm $||\cdot||$ we get the result. $\square$

Lemma 3.13. *Let r and β be small enough. Let $z^1, z^2 \in V(r,\beta)$ be different points such that $z^2 - z^1 \in S(\alpha)$ and $F(z^2), F(z^1) \in V(r,\beta)$. Then*

(1) $F(z^2) - F(z^1) \in S(\alpha)$,
(2) $||\pi^2(F(z^2) - F(z^1))|| \geqslant ||\pi^2(z^2 - z^1)||$.

Proof. By Lemma 3.12, $z^1 + t(z^2 - z^1) \in V(r, c\beta)$. Since DF is continuous we can write

$$F(z^2) - F(z^1) = \int_0^1 DF(z^1 + t(z^2 - z^1))(z^2 - z^1)\, dt$$
$$= \lim_{k \to \infty} \sum_{j=0}^{k} \frac{1}{k} DF\left(z^1 + \frac{j}{k}(z^2 - z^1)\right)(z^2 - z^1).$$

By Lemma 3.11, restricting β if necessary, $\sum_{j=0}^{k} \frac{1}{k} DF(z^1 + \frac{j}{k}(z^2 - z^1))(z^2 - z^1) \in S(\alpha)$ for all $k \in \mathbb{N}$ and hence the limit when $k \to \infty$ has to belong to $\overline{S(\alpha)} = S(\alpha)$.

For (2), using (1) and Lemma 3.12 applied to $F(z^1)$, $F(z^2)$, we get that $\psi(t) = (1-t)F(z^1) + tF(z^2) \in V(r, c\beta)$, for $t \in [0,1]$. Restricting β if necessary, by the mean value theorem, Lemma 3.10 and the definition of ψ, we have that

$$||\pi^2(z^2 - z^1)|| = ||\pi^2 \circ F^{-1} \circ \psi(1) - \pi^2 \circ F^{-1} \circ \psi(0)||$$
$$= \int_0^1 ||\pi^2 DF^{-1}(\psi(t))\psi'(t)||\, dt$$
$$\leqslant \int_0^1 ||\pi^2(\psi'(t))||\, dt = ||\pi^2(F(z^2) - F(z^1))||$$

and the statement holds. □

We will use the following result from degree theory. We will denote by $d(f, D, p)$ the degree of f at p relative to D. We recall that if $d(f, D, p) \neq 0$, then $p \in f(D)$. See [11] for details. We recall the following result.

Proposition 3.14. *Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be two continuous maps. If there exists a homotopy $H : [0,1] \times \bar{D} \to \mathbb{R}^n$ from f to g and $p \notin H(t, \partial D)$ for all $t \in [0,1]$ then $d(f, D, p) = d(g, D, p)$.*

Let $\mathscr{V}$ be an open neighborhood of $V(r,\beta) \backslash \{(0,0)\}$ such that $\mathscr{V} \cap \{x = 0\} = \emptyset$. Below D_γ^m will denote an open set of $\mathbb{R}^m$, such that $0 \in D_\gamma^m$ and that $\overline{D_\gamma^m}$ is homeomorphic to a closed ball. Therefore ∂D_γ^m will be homeomorphic to a sphere. Given $\gamma : D_\gamma^m \to \mathscr{V}$ we will denote by Γ the image of γ, i.e. $\Gamma = \gamma(D_\gamma^m)$. At some places we will identify γ with Γ. Let

$$H(\alpha) = \{\gamma : \overline{D_\gamma^m} \to \mathscr{V} : \gamma \in C^1, T_z\Gamma \subset S(\alpha)\ \forall z \in \Gamma \cap V(r,\beta), \gamma(\partial D_\gamma^m) \subset V(r,\beta)^c\}.$$

We note that the condition $T_z\Gamma \subset S(\alpha)$ implies that $\Gamma \cap V(r,\beta)$ can be expressed as the graph of a function $\psi : \pi^2(\Gamma \cap V(r,\beta)) \to \mathbb{R}^n$, in the form $\Gamma = \{(\psi(y), y) : y \in \Gamma \cap V(r,\beta)\}$ with

$$||D\psi(y)|| \leqslant 1/\alpha. \tag{3.9}$$

This is easily seen because if $v \in \mathbb{R}^m \backslash \{0\}$ we have that $t \mapsto (\psi(y+tv), y+tv)$ is a curve in Γ and hence its derivative at $t=0$, $(D\psi(y)v, v)$, belongs to $T_{(\psi(y),y)}\Gamma \subset S(\alpha)$ and then $||v|| \geqslant \alpha ||D\psi(y)v||$.

Our goal is to iterate manifolds of $H(\alpha)$ by F. A subtle and delicate point is to check that the iterates remain nonvoid. When $m=1$ this is a simple consequence of Bolzano's theorem, but if $m>1$ we are forced to apply degree theory. This motivates in part the definition of $H(\alpha)$.

Lemma 3.15. *If $c\beta < \alpha$, we have that if $\Gamma \in H(\alpha)$ then $F(\Gamma) \cap V(r,\beta) \in H(\alpha)$.*

Proof. We perform the change of variables C defined by $(x,v) \mapsto (x, y = v||x||)$ which transforms the cone-like domain $V(r,\beta)$ to the cylinder-like domain

$$\tilde{V}(r,\beta) = \{(x,v) \in \mathbb{R}^{n+m} : x \in V(r), ||v|| \leqslant \beta\}.$$

This change is invertible and its inverse is continuous when we restricted us to $\tilde{V}(r,\beta) \backslash \{x=0\}$. Indeed if $(x,y) \in V(r,\beta)$ then $x \neq 0$, and we can write the inverse change explicitly as $(x,y) \mapsto (x, v = y/||x||)$. In these new variables F is expressed as $\tilde{F} = C^{-1} \circ F \circ C$ with

$$\pi^1 \tilde{F}(x,v) = x + p(x, v||x||) + f(x, v||x||)$$

$$\pi^2 \tilde{F}(x,v) = \frac{v||x|| + q(x, v||x||) + g(x, v||x||)}{||x + p(x, v||x||) + f(x, v||x||)||}.$$

If $\Gamma \in H(\alpha)$, we denote by $\tilde{\Gamma}$ the image of Γ by this change of variables, i.e. $\tilde{\Gamma} = C^{-1}(\Gamma)$. We claim that $\tilde{\Gamma}$ can also be represented as a graph of a function $\tilde{\psi}$. Indeed, if $\Gamma = \{(\psi(y), y) : y \in D_\psi\}$, then $\tilde{\Gamma} = \{(\psi(y), y/||\psi(y)||) : y \in D_\psi\}$. Now, we are going to check that $\mathscr{X} : y \mapsto y/||\psi(y)||$ is invertible and that its inverse is continuous. First we note that $\psi \neq 0$ and then the map is well defined and continuous. Now we prove that $\mathscr{X}$ is one to one. If $y_1,\ y_2 \in D_\psi$ and $\mathscr{X}(y_1) = \mathscr{X}(y_2)$ we can write $y_1[||\psi(y_2)|| - ||\psi(y_1)||] + (y_1 - y_2)||\psi(y_1)|| = 0$ and then, if we assume that $y_1 \neq y_2$,

$$\frac{|\, ||\psi(y_2)|| - ||\psi(y_1)||\, |}{||y_1 - y_2||} = \frac{||\psi(y_1)||}{||y_1||}. \tag{3.10}$$

By (3.9)

$$\frac{|\, ||\psi(y_2)|| - ||\psi(y_1)||\, |}{||y_1 - y_2||} \leqslant \frac{||\psi(y_2) - \psi(y_1)||}{||y_1 - y_2||} \leqslant \frac{1}{\alpha}.$$

On the other hand, since $(\psi(y_1), y_1) \in V(r,\beta)$, $||\psi(y_1)||/||y_1|| \geqslant 1/\beta$. Putting these two last bounds in (3.10) we obtain $1/\beta \leqslant 1/\alpha$, which gives a contradiction.

Next we prove that $\mathscr{X}^{-1}$ is Lipschitz. Indeed, let $y_1,\ y_2 \in D_\psi$ and let $M_\psi > 0$ be such that for all $y \in D_\psi$, $||\psi(y)|| \leqslant M_\psi$. Such M_ψ exists by compactness of D_ψ. Then, using that $\alpha||\psi(y_1) - \psi(y_2)|| \leqslant ||y_1 - y_2||$ and $||y_i|| \leqslant \beta||\psi(y_i)||$ for

$i = 1, 2$, we get that

$$\begin{aligned}
\left\| \frac{y_1}{\|\psi(y_1)\|} - \frac{y_2}{\|\psi(y_2)\|} \right\| &= \frac{1}{\|\psi(y_1)\| \, \|\psi(y_2)\|} \| y_1(\|\psi(y_2)\| - \|\psi(y_1)\|) + (y_1 - y_2)\|\psi(y_1)\| \, \| \\
&\geqslant \frac{1}{\|\psi(y_1)\| \, \|\psi(y_2)\|} [\|y_1 - y_2\| \, \|\psi(y_1)\| - \|y_1\| \, | \, \|\psi(y_2)\| - \|\psi(y_1)\| \, |] \\
&\geqslant \frac{1}{\|\psi(y_1)\| \, \|\psi(y_2)\|} [\|\psi(y_1)\| - \|y_1\|/\alpha] \|y_1 - y_2\| \\
&\geqslant \frac{1}{\|\psi(y_1)\| \, \|\psi(y_2)\|} [\|\psi(y_1)\| - \beta\|\psi(y_1)\|/\alpha] \|y_1 - y_2\| \\
&= (1/M_\psi)[1 - \beta/\alpha] \|y_1 - y_2\|.
\end{aligned}$$

Therefore we can write $\tilde{\Gamma} = \{(\psi(\mathscr{X}^{-1}(v)), v) : v \in \mathscr{X}(D_\psi)\}$. We call $\tilde{\psi} = \psi \circ \mathscr{X}^{-1}$.

Now we look at the image of $\tilde{\Gamma} = \operatorname{graph} \tilde{\psi}$ by $\tilde{F}$. First we prove that the image of $B^m_\beta(0) = \{y \in \mathbb{R}^m : \|y\| \leqslant \beta\}$ by $\pi^2 \tilde{F} \circ (\tilde{\psi}(y), y)$ covers $B^m_\beta(0)$. For this we will use degree theory. Let

$$H(t, y) = \frac{y\|\tilde{\psi}(y)\| + tq(\tilde{\psi}(y), y\|\tilde{\psi}(y)\|) + tg(\tilde{\psi}(y), y\|\tilde{\psi}(y)\|)}{\|\tilde{\psi}(y) + tp(\tilde{\psi}(y), y\|\tilde{\psi}(y)\|) + tf(\tilde{\psi}(y), y\|\tilde{\psi}(y)\|)\|}$$

be a homotopy from the identity to $\pi^2 \tilde{F} \circ (\tilde{\psi}(y), y)$ and let $y_0 \in B^m_\beta(0)$. If $y_0 \in \partial B^m_\beta(0)$, then $(\tilde{\psi}(y_0), y_0) \in \partial \tilde{V}(r, \beta)$ and by the conclusions of Lemmas 3.7 and 3.8 translated to $\tilde{F}$ we deduce that $y_0 \notin H(t, \partial B^m_\beta(0))$ and hence from Proposition 3.14 we get that

$$d(\pi^2 \tilde{F} \circ (\tilde{\psi}, \mathrm{Id}), B^m_\beta(0), y_0) = d(\mathrm{Id}, B^m_\beta(0), y_0) = 1.$$

Going back to the original variables (x, y) we obtain that $F(\Gamma)$ is the image of

$$\gamma_1 = F \circ \gamma = C \circ (C^{-1} \circ F \circ C) \circ (C^{-1} \circ \gamma) = C \circ \tilde{F} \circ (C^{-1} \circ \gamma).$$

We will need to restrict the domain D_γ to D_{γ_1} in such a way that for all $\zeta \in D_{\gamma_1}$, $\gamma(\zeta) \in \mathscr{V}$. Therefore we also obtain that $F(\partial D_{\gamma_1}) \subset V(r, \beta)^c$. Finally the fact that $T_z(F(\Gamma)) \subset S(\alpha)$ for all $z \in F(\Gamma) \cap V(r, \beta)$ comes from Lemma 3.13. $\square$

With the previous lemmas we can prove Theorem 3.1.

Proof of the Theorem 3.1. Given $\Gamma \in H(\alpha)$ we define the sequence

$$\Gamma_0 = \Gamma, \quad \Gamma_k = F(\Gamma_{k-1}) \cap V(r, \beta), \quad k \geqslant 1.$$

By Lemma 3.15 all elements of this sequence belong to $H(\alpha)$. We introduce $I_k = F^{-k}(\Gamma_k)$. We claim that $(I_k)_k$ is a nested sequence of nonempty compact sets. Indeed:

$$I_k = F^{-k}(F(\Gamma_{k-1}) \cap V(r,\beta)) \subset F^{-k}(F(\Gamma_{k-1})) = F^{-(k-1)}(\Gamma_{k-1}) = I_{k-1}.$$

The fact that I_k are nonempty comes from Lemma 3.15. Hence $\bigcap_{k\geqslant 0} I_k \neq \emptyset$. Next we consider a particular sort of initial Γ_0. For every $x^0 \in V$ we define $\Gamma = \Gamma_0 = \{(x,y) : x = x^0, \|y\| \leqslant \beta\|x\|\}$. Let α given by Lemma 3.9. It is clear that $\Gamma \in H(\alpha)$ and that for all $z^1, z^2 \in \Gamma_0$, $z^2 - z^1 \in S(\alpha)$.

We will prove that $\bigcap_{k\geqslant 0} I_k$ reduces to a point. Assume that there exist $z^1, z^2 \in \bigcap_{k\geqslant 0} I_k$. Then $F^k(z^1), F^k(z^2) \in V(r,\beta)$, $\forall k \geqslant 0$. By Lemma 3.7 we have that $\|\pi^1(F^k(z^1))\|$ is a strictly decreasing sequence of real numbers. Therefore it has a limit which must be 0. Moreover, for all k, $\|\pi^2(F^k(z^1))\| \leqslant \beta\|\pi^1(F^k(z^1))\|$, thus $\pi^2(F^k(z^1))$ also goes to 0. The same happens to $\pi^2(F^k(z^2))$. Applying Lemma 3.13 iteratively we get

$$\|\pi^2(F^k(z^2) - F^k(z^1))\| \geqslant \|\pi^2(z^1 - z^2)\|.$$

Taking the limit when $k \to \infty$ we obtain $\pi^2(z^2) = \pi^2(z^1)$. Also, since $z^2 - z^1 \in S(\alpha)$, we have that $\pi^1(z^2) = \pi^1(z^1)$ and hence $z^2 = z^1$. Therefore $\bigcap_{k\geqslant 0} I_k$ is a point and has the form (x^0, y^0). Furthermore

$$\bigcap_{k\geqslant 0} I_k = \Gamma \cap \left\{(x,y) \in \mathbb{R}^{n+m} : \lim_{k\to\infty} F^k(z) = 0, F^k(z) \in V(r,\beta), k \geqslant 0\right\}.$$

We define φ by $\varphi(x^0) = y^0$. The graph of φ is the invariant manifold we looked for. Now it remains to be proved that φ is Lipschitz. If we assume that $\mathrm{Lip}\,\varphi$ is not smaller than α, there would exist two different points $x^1, x^2 \in V(r,\beta)$ such that

$$\frac{\|\varphi(x^2) - \varphi(x^1)\|}{\|x^2 - x^1\|} \geqslant \alpha.$$

Applying Lemma 3.13 iteratively we have

$$\|\pi^2(F^k(x^2, \varphi(x^2)) - F^k(x^1, \varphi(x^1)))\| \geqslant \|\varphi(x^2) - \varphi(x^1)\| \geqslant \alpha\|x^2 - x^1\|.$$

Since $(x^1, \varphi(x^1))$ and $(x^2, \varphi(x^2))$ belong to the stable manifold

$$\lim_{k\to\infty} \pi^2(F^k(x^2, \varphi(x^2)) - F^k(x^1, \varphi(x^1))) = 0$$

and hence we deduce that $x^2 = x^1$, which is a contradiction. Therefore φ is Lipschitz with $\mathrm{Lip}\,\varphi < \alpha$. $\square$

Remark 3.16. From the fact that we can take α as small as we want if we take r small enough, we get that φ has an arbitrarily small Lipschitz constant in a sufficiently small neighborhood of the origin. Therefore φ is differentiable at 0 and $D\varphi(0) = 0$.

4. The analytic case

In this section we shall prove that if F is analytic then φ is also analytic in a suitable complex enlargement of its domain. We consider F defined in an open set of $\mathbb{C}^{n+m}$. We introduce the following notation: if $x \in \mathbb{C}^n$, $\|\mathrm{Re}\, x\|$ and $\|\mathrm{Im}\, x\|$ mean the norm of $(\mathrm{Re}\, x_1, \ldots, \mathrm{Re}\, x_n)$ and $(\mathrm{Im}\, x_1, \ldots, \mathrm{Im}\, x_n)$ respectively as elements of $\mathbb{R}^n$ and

$$\|x\| = \max\{\|\mathrm{Re}\, x\|, \|\mathrm{Im}\, x\|\}.$$

We take the norm of $\|y\|$ in an analogous way and finally $\|(x,y)\| = \max\{\|x\|, \|y\|\}$.

Given $\gamma, r > 0$ we define the sets

$$\Omega(r,\gamma) = \{x \in \mathbb{C}^n : \ \mathrm{Re}\, x \in V(r), \|\mathrm{Im}\, x\| < \gamma \|\mathrm{Re}\, x\|\},$$

$$\Lambda(r,\gamma,\beta) = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m : x \in \Omega(r,\gamma), \|y\| \leqslant \beta \|x\|\}.$$

We will need the set $\Omega(r,\gamma)$ to be invariant by $x \mapsto \pi^1 F(x,y)$ for $\|y\| \leqslant \beta \|x\|$. Actually we will need that there exists an invariant open set containing $V(r)$ and contained in $\Omega(r,\gamma)$. We will see in Lemma 4.3 that a technical sufficient condition for the invariance of $\Omega(r,\gamma)$ for some r, γ is

H4 For all $x \in V^1(\rho)$, $\|\mathrm{Id} + D_x p(x,0)\| + \|\mathrm{Id} - \frac{1}{N_p} D_x p(x,0)\| < 2$.

Note that if H4 holds, since $V^1(\rho)$ is a compact set, there exists $\eta > 0$ such that for all $x \in V^1(\rho)$ we have $\|\mathrm{Id} + D_x p(x,0)\| + \|\mathrm{Id} - \frac{1}{N_p} D_x p(x,0)\| \leqslant 2 - \eta$.

Theorem 4.1. *Let F be an analytic map of the form* (3.1). *Assume that the hypotheses* H1–H4 *hold. Then, the map φ obtained in Theorem* 3.1 *is analytic in $V(r)$.*

To prove Theorem 4.1 we will consider a suitable analytic initial function and then the sequence of its iterates by the graph transform.

Since we are interested in the stable manifold we will consider the graph transform associated to the inverse map F^{-1}. This causes that when one has an iterate, next iterate is defined implicitly. Rouché's theorem is used to show that the graph of the next iterate has no irregular points and the implicit function theorem implies analyticity. Lemma 4.3 below provides the necessary estimates to apply Rouché's theorem.

First we state a technical lemma.

Lemma 4.2. *Let* $A(x,y) = (a_{ij}(x,y))_{i,j}$ *be a* $k \times k$ *matrix such that*

(a) *For all* $i,j \in \{1,\dots,k\}$, $a_{i,j} : \Lambda(r,\gamma,\beta) \to \mathbb{C}^k$ *are* C^1 *homogeneous functions of degree* $N-1$.
(b) *There exists a constant* $M > 0$ *such that the matrix* $A(x,0)$ *satisfies*

$$||\mathrm{Id} + A(x,0)|| \leqslant 1 - M||x||^{N-1}, \quad \textit{for all } x \in V(r).$$

Then, there exist positive numbers $r, \beta, \gamma, K > 0$ *such that*

$$||\mathrm{Id} + A(x,y)|| \leqslant 1 - K||x||^{N-1}, \quad \textit{for all } x \in \Lambda(r,\gamma,\beta). \tag{4.1}$$

Proof. We denote $x = x_1 + ix_2$ and $A(x,0) = A_1(x_1,x_2) + iA_2(x_1,x_2)$. If we take $\gamma < 1$ and $x \in \Omega(r,\gamma)$ then $||x|| = ||x_1||$. Moreover, it is clear that there exists $K_1 > 0$ such that $\max_{||w|| \leqslant \gamma ||x_1||} ||D_{x_2} A_1(x_1,w)|| \leqslant K_1 ||x||^{N-2}$. Then, if $x \in \Omega(r,\gamma)$, by hypothesis (b)

$$\begin{aligned} ||\mathrm{Id} + A_1(x_1,x_2)|| &\leqslant ||\mathrm{Id} + A_1(x_1,0)|| + ||A_1(x_1,0) - A_1(x_1,x_2)|| \\ &\leqslant 1 - (M - \gamma K_1)||x||^{N-1} \\ &\leqslant 1 - M_1 ||x||^{N-1} \end{aligned}$$

with $M_1 > 0$ if we take γ small enough. Moreover, since $A_2(x_1,0) = 0$, there exists $K_2 > 0$ such that $||A_2(x_1,x_2)|| \leqslant \gamma K_2 ||x||^{N-1}$. Let $v \in \mathbb{C}^k$ be such that $||v|| = 1$. We write $v = v_1 + iv_2$. Using the previous bounds, if r and γ are small enough, there exists $M_0 > 0$ such that

$$||v_1 + A_1(x_1,x_2)v_1 - A_2(x_1,x_2)v_2|| \leqslant 1 - M_0||x||^{N-1}$$

and

$$||v_2 + A_1(x_1,x_2)v_2 + A_2(x_1,x_2)v_1|| \leqslant 1 - M_0||x||^{N-1}.$$

Therefore, $||\mathrm{Id} + A(x,0)|| \leqslant 1 - M_0||x||^{N-1}$ for $x \in \Omega(r,\gamma)$.

Finally, using again the mean value theorem

$$\begin{aligned} ||\mathrm{Id} + A(x,y)|| &\leqslant ||\mathrm{Id} + A(x,0)|| + ||A(x,0) - A(x,y)|| \\ &\leqslant 1 - (M_0 - \beta K_3)||x||^{N-1} \end{aligned}$$

which implies the bound (4.1) if β is small enough. $\square$

Lemma 4.3. *There exist* $r, \gamma, \beta > 0$ *such that*

(1) *if* $(x, y) \in A(r, \gamma, \beta)$ *then* $\pi^1 F(x, y) \in \Omega(r, \gamma)$,
(2) *if* $(x, y) \in A(r, \gamma, \beta)$ *and* $||y|| = \beta||x||$ *then* $||\pi^2 F(x, y) - y|| < ||y||$,
(3) *if* $(x, y) \in A(r, \gamma, \beta)$ *and* $||y|| = \beta||x||$ *then* $\beta||\pi^1 F(x, y)|| < ||\pi^2 F(x, y)||$.

Proof. (1) Let $x \in \Omega(r, \gamma)$. We write $x = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}^n$ and $p(x, 0) = p^1(x_1, x_2) + ip^2(x_1, x_2)$ with $p^1(x_1, x_2), p^2(x_1, x_2) \in \mathbb{R}^n$. By the Cauchy–Riemann equations, we have that

$$D_{x_1} p^1 = D_{x_2} p^2, \quad D_{x_2} p^1 = -D_{x_1} p^2. \tag{4.2}$$

We observe that, since $p^2(x_1, 0) = 0$, by (4.2) we get

$$D_{x_2} p^1(x_1, 0) = -D_{x_1} p^2(x_1, 0) = 0. \tag{4.3}$$

We claim that there exist positive constants γ_0, K_0 such that

$$\begin{aligned} &||x_2 + p^2(x_1, x_2)|| - \gamma||x_1 + p^1(x_1, x_2)|| \leqslant -\gamma K_0 ||x||^{N_p} \\ &\text{for all } x \in \Omega(r, \gamma), \quad \gamma < \gamma_0. \end{aligned} \tag{4.4}$$

Indeed, we denote

$$C(x_1, x_2) \equiv \int_0^1 D_{x_2} p^2(x_1, s x_2)\, ds, \tag{4.5}$$

$$A(x_1, x_2) \equiv \frac{1}{N_p} D_{x_1} p^1(x_1, x_2), \tag{4.6}$$

$$B(x_1, x_2) \equiv \frac{1}{N_p} D_{x_2} p^1(x_1, x_2) \tag{4.7}$$

and we notice that, by (4.2) and (4.3)

$$C(x_1, 0) = N_p A(x_1, 0), \quad B(x_1, 0) = 0.$$

Let $x \in \Omega(r, \gamma)$. Then

$$\begin{aligned} ||x_2 + p^2(x_1, x_2)|| &= ||(\mathrm{Id} + C(x_1, x_2))x_2|| \leqslant ||x_2|| \; ||\mathrm{Id} + C(x_1, x_2)|| \\ &\leqslant \gamma||x_1|| \; ||\mathrm{Id} + C(x_1, x_2)|| \end{aligned} \tag{4.8}$$

and, by Euler's theorem

$$\begin{aligned} ||x_1 + p^1(x_1, x_2)|| &= ||(\mathrm{Id} + A(x_1, x_2))x_1 + B(x_1, x_2)x_2|| \\ &\geqslant ||x_1|| \left(\frac{1}{||(\mathrm{Id} + A(x_1, x_2))^{-1}||} - \gamma||B(x_1, x_2)|| \right). \end{aligned} \tag{4.9}$$

Next we will see that there exist γ small enough and K_1 such that for all $(x_1, x_2) \in \Omega(r, \gamma)$ we have that

$$||\mathrm{Id} + C(x_1, x_2)|| + ||(\mathrm{Id} + A(x_1, x_2))^{-1}|| < 2 - K_1 ||x_1||^{N_p - 1}. \tag{4.10}$$

Bound (4.10) is a consequence of hypothesis H4. Indeed,

$$\begin{aligned}
&||\mathrm{Id} + C(x_1, 0)|| + ||\mathrm{Id} - A(x_1, 0)|| \\
&\quad = ||\mathrm{Id} + N_p A(x_1, 0)|| + ||\mathrm{Id} - A(x_1, 0)|| \\
&\quad = ||\mathrm{Id} + N_p ||x_1||^{N_p-1} A(x_1/||x_1||, 0)|| + ||\mathrm{Id} - ||x_1||^{N_p-1} A(x_1/||x_1||, 0)|| \\
&\quad \leqslant ||x_1||^{N_p-1}[||\mathrm{Id} + N_p A(x_1/||x_1||, 0)|| - 1 + 1/||x_1||^{N_p-1} \\
&\qquad + ||\mathrm{Id} - A(x_1/||x_1||, 0)|| - 1 + 1/||x_1||^{N_p-1}] \\
&\quad = 2 + ||x_1||^{N_p-1}[||\mathrm{Id} + N_p A(x_1/||x_1||, 0)|| + ||\mathrm{Id} - A(x_1/||x_1||, 0)|| - 2] \\
&\quad \leqslant 2 - \eta ||x_1||^{N_p-1}. \qquad (4.11)
\end{aligned}$$

Using (4.11), the mean value theorem and the homogeneity of the derivatives of C and A, there exists γ small enough such that

$$\begin{aligned}
&||\mathrm{Id} + C(x_1, x_2)|| + ||\mathrm{Id} - A(x_1, x_2)|| \\
&\quad \leqslant ||\mathrm{Id} + C(x_1, 0)|| + ||\mathrm{Id} - A(x_1, 0)|| \\
&\qquad + ||C(x_1, 0) - C(x_1, x_2)|| + ||A(x_1, 0) - A(x_1, x_2)|| \\
&\quad \leqslant 2 - \eta ||x_1||^{N_p-1} + 2\gamma K_1 ||x_1||^{N_p-1} \\
&\quad \leqslant 2 - (\eta/2)||x_1||^{N_p-1}.
\end{aligned}$$

This implies (4.10). Using the general simple fact that if $a, b \geqslant 0$ and $a + b \leqslant 2 - \eta$ then $ab \leqslant (1 - \eta/2)^2$, from (4.10) we obtain the following bound for the product of norms $||\mathrm{Id} + C(x_1, x_2)||\, ||(\mathrm{Id} + A(x_1, x_2))^{-1}|| \leqslant 1 - K_2 ||x_1||^{N_p-1}$. We observe that, if γ is small enough, then, by hypothesis H1

$$||(\mathrm{Id} + A(x_1, x_2))^{-1}|| \geqslant \frac{1}{||\mathrm{Id} + A(x_1, x_2)||} > 1$$

and that, there exists $K_3 > 0$ such that $\max_{||w|| \leqslant \gamma ||x_1||} ||B(x_1, w)|| \leqslant \gamma K_3 ||x_1||^{N_p-1}$. Therefore, if γ is small enough,

$$\begin{aligned}
||\mathrm{Id} + C(x_1, x_2)|| &\leqslant \frac{1 - K_2 ||x_1||^{N_p-1}}{||(\mathrm{Id} + A(x_1, x_2))^{-1}||} \\
&\leqslant \frac{1}{||(\mathrm{Id} + A(x_1, x_2))^{-1}||} - K_2 ||x_1||^{N_p-1} - \gamma ||B(x_1, x_2)|| + \gamma ||B(x_1, x_2)|| \\
&\leqslant \frac{1}{||(\mathrm{Id} + A(x_1, x_2))^{-1}||} - \gamma ||B(x_1, x_2)|| - K_4 ||x_1||^{N_p-1}
\end{aligned}$$

which together with (4.8) and (4.9) implies (4.4).

To prove (1) we have to check that $\mathrm{Re}\,\pi^1 F(x,y) \in V(r,\gamma)$. We have that

$$\mathrm{Re}\,\pi^1 F(x,y) = [x_1 + p^1(x_1,0)] + [\mathrm{Re}\,p(x,0) - p^1(x_1,0)]$$

$$+\ \mathrm{Re}\,(p(x,y) - p(x,0)) + \mathrm{Re}\,f(x,y). \tag{4.12}$$

The second term in the right hand of (4.12) is less than $\gamma K\|x_1\|^{N_p-1}$. The third term is bounded by $\max_{\|w\|\leqslant\beta\|x\|} \|D_y \mathrm{Re}\,p(x,w)\|\ \|y\| \leqslant \beta K\|x\|^{N_p}$ and the fourth term is $o(\|x\|^{N_p})$. Therefore, by H3, if γ and β are sufficiently small $\mathrm{Re}\,\pi^1 F(x,y) \in V(r)$.

It remains to prove that $\|\mathrm{Im}\,\pi^1 F(x,y)\| < \gamma\|\mathrm{Re}\,\pi^1 F(x,y)\|$. We have that

$$\begin{aligned}
&\|\mathrm{Im}\,(\pi^1 F(x,y))\| - \gamma\|\mathrm{Re}\,(\pi^1 F(x,y))\| \\
&\leqslant \|x_2 + p^2(x_1,x_2)\| - \gamma\|x_1 + p^1(x_1,x_2)\| \\
&\quad +\ \|p^2(x_1,x_2) - \mathrm{Im}\,p(x,y)\| + \gamma\|p^1(x_1,x_2) - \mathrm{Re}\,p(x,y)\| \\
&\quad +\ \|\mathrm{Im}\,f(x,y)\| + \gamma\|\mathrm{Re}\,f(x,y)\| \\
&\leqslant -K_0\gamma\|x\|^{N_p} + 2\beta K\|x\|^{N_p} + o(\|x\|^{N_p}) < 0
\end{aligned}$$

if r and β are small.

(2) Let $x \in \Omega(r,\gamma)$ and y such that $\|y\| = \beta\|x\|$. Let $j \in \{1,\dots,m\}$ be such that $|y_j| = \|y\|$. Then $\|\pi^2 F(x,y) - y\| = \|q(x,y) + g(x,y)\| \leqslant K\|x\|^{N_q} = K\|x\|^{N_q-1}\|y\|/\beta < \|y\|$ if r is such that $Kr^{N_q-1} < \beta$, since, as $\|y\| = \beta\|x\|$, $y \neq 0$.

(3) We will see that under the conditions in (3)

$$\|y + q(x,y) + g(x,y)\| > \|y\| \tag{4.13}$$

and

$$\|x + p(x,y) + f(x,y)\| < \|x\|. \tag{4.14}$$

From 4.13 and 4.14, (3) follows immediately. We deal with (4.13). Since $D_x q(x,0) = 0$, we have that $q(x,0) = 0$. Moreover $q(x,y) = Q(x,y)y$ where $Q(x,y) = \int_0^1 D_y q(x,sy)\,ds$. Clearly $Q(x,0) = D_y q(x,0)$, then by (3.3) it is clear that the matrix $-Q(x,y)$ satisfies the hypotheses of Lemma 4.2, therefore, there exist $\beta, K_0 > 0$ such that, if r,γ are small enough, for all $(x,y) \in \Lambda(r,\gamma,\beta)$, we have $\|\mathrm{Id} - Q(x,y)\| \leqslant 1 - K_0\|x\|^{N_q-1}$. Thus, we have that $\|(\mathrm{Id} + Q(x,y))^{-1}\| \leqslant 1 - M_0\|x\|^{N_q-1}$ for some constant M_0.

If r is small enough, then $\max_{||w|| \leqslant ||x||} ||g(x,w)|| \leqslant \beta(M_0/2)||x||^{N_q}$, and since $||y|| = \beta||x||$,

$$\begin{aligned} ||y + q(x,y) + g(x,y)|| &\geqslant ||(\mathrm{Id} + Q(x,y))y|| - ||g(x,y)|| \\ &\geqslant \frac{1}{||(\mathrm{Id} + Q(x,y))^{-1}||}||y|| - \frac{M_0}{2}||y||\ ||x||^{N_q-1} \\ &\geqslant \left(1 + \frac{M_0}{2}||x||^{N_q-1}\right)||y|| \\ &> ||y||, \end{aligned}$$

since, as $||y|| = \beta||x||$, $y \neq 0$.

Using Euler's theorem and (3.2) we can prove bound (4.14) in a similar way. □

We will also need a multidimensional version of the classical Rouché's theorem. First we recall the definitions of index and multiplicity.

Definition 4.4. Let $D \subset \mathbb{C}^n$ be an open set and f a continuous function on $\bar{D}$. Let z_0 be an isolated zero of f.

(1) We define the index of z_0 as $i(f, z_0, 0) = d(f, \mathscr{U}, 0)$ where $\mathscr{U}$ is any bounded neighborhood of z_0 which does not contain any zero of f different from z_0 and d stands for topological degree.
(2) We define the multiplicity of z_0 as $i(f, x_0, p)$. We say that z_0 is simple if its multiplicity is one.

The following version of Rouché's theorem can be found in [11].

Theorem 4.5. *Let D be a bounded, open set in $\mathbb{C}^n$. Suppose that f, g are two holomorphic functions on $\bar{D}$ such that $||g(z)|| < ||f(z)||$ for all $z \in \partial D$. Then, f has finitely many zeros in D, and, counting multiplicity, f and $f + g$ have the same number of zeros in D.*

In particular, if f has a unique zero in D of multiplicity one, $f + g$ also has a unique zero in D.

We define the set of functions

$$\mathscr{H} = \{h : \Omega(r,\gamma) \to \mathbb{C}^m : h \text{ real analytic in } \Omega, ||h(x)|| \leqslant \beta||x||\}$$

and also the sets

$$\Lambda^0 = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m : x \in \Omega(r,\gamma), ||y|| < \beta||x||\},$$

$$\Lambda = \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^m : x \in \Omega(r,\gamma), ||y|| \leqslant \beta||x||\},$$

$$D(x_0) = \{z \in \mathbb{C}^n : ||z|| < \beta||x_0||\} \quad \text{for } x_0 \in \Omega(r,\gamma).$$

For $x_0 \in \Omega(r,\gamma)$, $y \in D(x_0)$ and $h \in \mathscr{H}$, we define

$$H(x_0, y) = \pi^2 F(x_0, y) - h(\pi^1 F(x_0, y))$$

and we want to solve $H(x_0, y) = 0$ with respect to y. The interpretation of $H(x_0, y) = 0$ is that, if we solve $y = y^*(x_0)$, graph y^* is the preimage by F of graph h. Notice that if $x_0 \in \Omega(r,\gamma)$ and $y \in D(x_0)$, H is well defined and analytic in Λ^0. Let us see that $H(x_0, y) = 0$ has a unique solution in $D(x_0)$. Indeed, by Lemma 4.3, if $x_0 \in \Omega(r,\gamma)$ and $||y|| = \beta||x_0||$ then

$$||\pi^2 F(x_0, y) - y|| < ||y||.$$

Therefore by Rouché's theorem, the functions y and $\pi^2 F(x_0, y)$ (as functions of y) have the same number of zeros in $D(x_0)$. Since the first function is the identity they have a unique zero.

On the other hand, if $||y|| = \beta||x_0||$, by Lemma 4.3 we have that

$$\beta||\pi^1 F(x_0, y)|| < ||\pi^2 F(x_0, y)||$$

and hence

$$\begin{aligned} ||H(x_0, y) - \pi^2 F(x_0, y)|| &= ||h(\pi^1 F(x_0, y))|| \leqslant \beta||\pi^1 F(x_0, y)|| \\ &< ||\pi^2 F(x_0, y)|| \end{aligned}$$

and again by Rouché's theorem, H has a unique zero in $D(x_0)$ which we denote by $y^*(x_0)$. Clearly $||y^*(x_0)|| < \beta||x_0||$.

By the implicit function theorem, since this zero is unique, it depends analytically with respect to x_0. Hence we can define a map $\mathscr{F} : \mathscr{H} \to \mathscr{H}$ by

$$\mathscr{F}h(x) = y^*(x),$$

where $y^*(x)$ such that $H(x, y^*(x)) = 0$ for all $x \in \Omega(\delta)$.

Since H is real analytic and the solution $y(x)$ is unique the latter must be real analytic. Otherwise the conjugate would be another solution on $D(x)$. We have proved that $\mathscr{F}$ sends $\mathscr{H}$ into $\mathscr{H}$.

Furthermore by construction we have $F(\text{graph}(\mathscr{F}h)) \subset \text{graph}(h)$ and if $0 \leqslant m \leqslant n$

$$F^m(\text{graph}(\mathscr{F}^n h)) \subset \text{graph}(\mathscr{F}^{n-m} h) \in \Lambda.$$

Given $h_0 \in \mathscr{H}$ we define the sequence $h_n = \mathscr{F}^n h_0 \in \Lambda$. Since $h_n \in \mathscr{H}$ the sequence is uniformly bounded and, by Montel's theorem, it has a subsequence convergent to some function $h \in \mathscr{H}$. To check that $F^m(\text{graph}(h)) \in \Lambda$, we shall assume the contrary, that is, that there exist $m \geqslant 0$ and $x \in \Omega\ (r,\gamma)$ such that $F^m(x, h(x)) \notin \Lambda$. Since F^m is continuous there exists $\varepsilon > 0$ such that if $||y - h(x)|| < \varepsilon$ then $F^m(x, y) \notin \Lambda$, but for $n > m$ big enough $||h_n(x) - h(x)|| < \varepsilon$, and this would imply $F^m(x, h_n(x)) \notin \Lambda$ which is a contradiction. Hence $F^m(\text{graph}(h)) \in \Lambda$, $\forall m \in \mathbb{N}$.

If $x \in \Omega(r,\gamma) \cap \mathbb{R}^n = V$ we have, if β is small enough, that

$$\operatorname{graph} h_{|V} \subset W^s_{V,r} \cap \{y \in \mathbb{R}^m : ||y|| \leqslant \beta ||x||\} = \operatorname{graph} \varphi.$$

Therefore, $h_{|V} = \varphi$ which implies that φ is a real analytic function. This ends the proof of Theorem 4.1.

5. Examples

1. A simple example of application of the above theorem is the map $F : \mathbb{R}^{2+1} \to \mathbb{R}^{2+1}$ defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_1^3 + 3x_1x_2^2 + f_1(x_1,x_2,y) \\ x_2 + x_2^3 - 3x_1^2x_2 + f_2(x_1,x_2,y) \\ y + q(x_1,x_2,y) + g(x_1,x_2,y) \end{pmatrix}$$

where $q(x_1,x_2,y)$ is a homogeneous polynomial of degree 3, f_1, f_2 and g are analytic functions of order 4. We will work with the supremum norm. Let $r < 1/\sqrt{3}$, $V(r) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in (0,r), 5|x_2| < |x_1|\}$, $\rho = 1/\sqrt{3}$ and $V^1(\rho) = \{(1/\sqrt{3}, x_2) : |x_2| \leqslant 1/(5\sqrt{3})\}$. We assume that $q(x_1,x_2,y) = y\tilde{q}(x_1,x_2,y)$ and $\tilde{q}(x_1,x_2,0) > 0$ on $V^1(\rho)$. Below we will check that F satisfies the hypotheses of Theorem 4.1. Therefore, if r is small there exists a stable invariant manifold of the origin given by the graph of an analytic function $\varphi : V(r) \to \mathbb{R}$.

We write $p_1(x_1,x_2,y) = -x_1^3 + 3x_1x_2^2$ and $p_2(x_1,x_2,y) = x_2^3 - 3x_1^2x_2$ and $p = (p_1,p_2)$. The hypothesis H1 is equivalent to $3x_1^2 - 3x_2^2 > |6x_1x_2|$ for $(x_1,x_2) \in V^1(\rho)$ which is easily seen to be true. The hypothesis H2 holds by the conditions we have on q.

To check H3, given $x = (x_1,x_2) \in V(r)$ we estimate the distances of $x + p(x,0)$ to the three parts of $\partial V(r)$:

$$\{(x_1,x_2) : x_1 - 5x_2 = 0,\ 0 \leqslant x_1 \leqslant r\},$$

$$\{(x_1,x_2) : x_1 + 5x_2 = 0,\ 0 \leqslant x_1 \leqslant r\} \quad \text{and}$$

$$\{(x_1,x_2) : x_1 = r,\ |x_2| \leqslant r/5\}.$$

Since $x_1 - 5x_2 > 0$ and $x_1 > 0$ we have that

$$\begin{aligned} \operatorname{dist}(x + p(x,0), X_1 - 5X_2 = 0) &= \frac{x_1 - x_1^3 + 3x_1x_2^2 - 5(x_2 + x_2^3 - 3x_2x_1^2)}{\sqrt{26}} \\ &= \frac{x_1(1 - x_1^2 + 3x_2^2) - 5x_2(1 + x_2^2 - 3x_1^2)}{\sqrt{26}} \\ &\geqslant \frac{x_1(2x_1^2 + 2x_2^2)}{\sqrt{26}} \geqslant \frac{2}{\sqrt{26}} x_1^3 \end{aligned}$$

which means that $x + p(x, 0)$ stays at the same side of $X_1 - 5X_2 = 0$ than (x_1, x_2) does and that the distance is $O(||(x_1, x_2)||^3)$.

Analogously $\operatorname{dist}(x + p(x, 0), X_1 + 5X_2 = 0) \geqslant (2/\sqrt{26})x_1^3$. Since $-x_1 + r > 0$, $x_1 > 0$ and $5|x_2| < x_1$

$$\begin{aligned}\operatorname{dist}(x + p(x, 0), -X_1 + r = 0) &= -x_1 + x_1^3 - 3x_1x_2^2 + r \geqslant x_1(x_1^2 - 3x_2^2)\\ &\geqslant \frac{22}{25}x_1^3 > 0.\end{aligned}$$

This proves that if $x \in V(r)$, $\operatorname{dist}(x + p(x, 0), V(r)^c) > (2/\sqrt{26})x_1^3 = (2/\sqrt{26})||x||^3$.

Finally, hypothesis H4 follows directly from

$$||\mathrm{Id} + D_x p(x, 0)|| + \left|\left|\mathrm{Id} - \frac{1}{3}D_x p(x, 0)\right|\right| < 2 - 2x_1^2 + 2x_2^2 + 8x_1|x_2| < 2$$

for $x = (x_1, x_2) \in V^1(\rho)$.

2. The second example is the elliptic three-body problem. It consists in the study of the motion of three bodies of masses $1 - \mu$, μ, 0, with $\mu \in (0, 1)$. The first two bodies, called primaries, move on ellipses of eccentricity e and semimajor axis a in a plane. The third body moves in the space under the effect of the attraction of the two primaries. The formulae $\zeta_1 = (z_1 \cos f, z_1 \sin f, 0)$, $\zeta_2 = -(z_2 \cos f, z_2 \sin f, 0)$ with

$$z_1 = \frac{\mu(1 - e^2)}{1 + e\cos f}, \quad z_2 = \frac{(1 - \mu)(1 - e^2)}{1 + e\cos f} \quad \text{and} \quad \frac{df}{dt} = \frac{(1 + e\cos f)^2}{(1 - e^2)^{3/2}} \tag{5.1}$$

describe the position of the primaries. The motion of the third body is governed by the equation

$$\ddot{\zeta} = -(1 - \mu)\frac{\zeta - \zeta_1}{R_1^3} - \mu\frac{\zeta - \zeta_2}{R_2^3}, \quad \zeta = (X, Y, Z),$$

where $R_1 = ||\zeta - \zeta_1||$ and $R_2 = ||\zeta - \zeta_2||$. To study the behavior of the system in a vicinity of infinity, we perform a change of coordinates, inspired in the McGehee coordinates, to transform the infinity to a suitable manifold. This submanifold will be foliated by periodic orbits which will be labeled by two parameters, α_∞ and ρ_∞. We are interested in the invariant manifolds of these periodic orbits.

We introduce the new coordinates $x, y, \alpha, \rho, \theta, \tau$ given by

$$X = \frac{2}{x^2}\cos\alpha\cos\theta, \quad Y = \frac{2}{x^2}\sin\alpha\cos\theta, \quad Z = \frac{2}{x^2}\sin\theta$$

and

$$\dot{X} = y\cos\alpha\cos\theta - x^2\rho\sin\alpha\cos\theta - \tau\theta\cos\alpha\sin\theta,$$

$$\dot{Y} = y\sin\alpha\cos\theta + x^2\rho\cos\alpha\cos\theta - \tau\theta\sin\alpha\sin\theta,$$

$$\dot{Z} = y\sin\theta + \tau\theta\cos\theta.$$

Writing the equations in these new variables we get the system

$$\begin{aligned}
&\dot{x} = -\frac{1}{4}x^3 y, \quad \dot{y} = -\frac{1}{4}x^4 + O(x^6) + O(x^2\theta^2\tau^2),\\
&\dot{\tau} = -\frac{1}{2}x^2\tau^2 + O(x^8), \quad \dot{\theta} = \frac{1}{2}x^2\theta\tau,\\
&\dot{\alpha} = \frac{1}{2}x^4\rho, \quad \dot{\rho} = \theta^2\tau x^2\rho + O(x^6) + O(\theta^4\tau x^2\rho),\\
&\dot{f} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}
\end{aligned} \tag{5.2}$$

if θ is close to zero. We observe that the points of the form $(x, y, \tau, \theta, \alpha, \rho) = (0, 0, 0, 0, \alpha_\infty, \rho_\infty)$ for every fixed constants α_∞ and ρ_∞, give place to periodic orbits. The set

$$I = \{(x, y, \alpha, \rho, \theta, \tau, f) : x = 0\}$$

is called the infinity manifold, and obviously, it is invariant. Moreover, the flow extends analytically to it. The set $I_0 = I \cap \{y = 0, \tau = 0, \theta = 0\}$ is called the parabolic infinity. It is foliated by periodic orbits which can be labeled by α and ρ. Our objective is to prove that they have an analytic stable invariant manifold.

For this we perform the change of coordinates given by

$$a = \frac{\alpha + 2y\rho - \alpha_\infty}{x}, \quad r = \frac{\rho(1-\theta^2) - \rho_\infty}{x},$$

then, we get the system

$$\begin{aligned}
&\dot{x} = -\frac{1}{4}x^3 y, \quad \dot{y} = -\frac{1}{4}x^4 + \text{h.o.t.},\\
&\dot{\tau} = -\frac{1}{2}x^2\tau^2 + \text{h.o.t.}, \quad \dot{\theta} = \frac{1}{2}x^2\theta\tau,\\
&\dot{a} = \frac{1}{4}x^2 ya + \text{h.o.t.}, \quad \dot{r} = \frac{1}{4}x^2 yr + \text{h.o.t.}\\
&\dot{f} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}.
\end{aligned}$$

Computing the Poincaré map P from $f=0$ to 2π we obtain

$$x_1 = x - Kx^3y + \text{h.o.t.}, \quad y_1 = y - Kx^4 + \text{h.o.t.},$$
$$\tau_1 = \tau - 2Kx^2\tau^2 + \text{h.o.t.}, \quad \theta_1 = \theta + 2Kx^2\tau\theta + \text{h.o.t.},$$
$$a_1 = a + Kx^2ya + \text{h.o.t.}, \quad r_1 = r + Kx^2ya + \text{h.o.t.},$$

where $(x_1, y_1, \theta_1, \tau_1, a_1, r_1) = P(x, y, \theta, \tau, a, r)$ and

$$K = \frac{(1-e^2)^{3/2}}{4}\int_0^{2\pi}\frac{1}{(1+e\cos f\,)^2}\,df = \frac{\pi}{2}.$$

It is clear that the origin is a fixed point of the Poincaré map, which corresponds to the periodic orbit with parameters α_∞ and ρ_∞. This map is not yet in a suitable form. We perform the change of variables given by $u = x + y,\ v = x - y$, and then, the Poincaré map takes the form

$$u_1 = u - Cu(u+v)^3 + \text{h.o.t.}, \quad v_1 = v + Cv(u+v)^3 + \text{h.o.t.},$$
$$\tau_1 = \tau - 4C(u+v)^2\tau^2 + \text{h.o.t.}, \quad \theta_1 = \theta + 4C(u+v)^2\tau\theta + \text{h.o.t.},$$
$$a_1 = a + C(u+v)^2(u-v)a + \text{h.o.t.}, \quad r_1 = r + C(u+v)^2(u-v)r + \text{h.o.t.},$$

where $C = K/8$.

This map satisfies the hypotheses H1–H4 of Theorem 4.1. It is sufficient to consider the convex set

$$V = \{(u,\tau) : c\tau < u < \tau/c\},$$

for some $c>0$ fixed. To check them is a straightforward calculation. Therefore from Theorem 4.1 we conclude that there exists a two-dimensional stable invariant manifold of the origin, which, for r_0 small enough, can be expressed as the graph of an analytic function $(v,\theta,r,a) = \varphi(u,\tau),\ (u,\tau)\in V(r_0)$.

It remains to transform the invariant manifold to the originals coordinates. On the invariant manifold, $x = (u+v)/2 = (u+\varphi_1(u,\tau))/2 = h(u,\tau)$. We observe that h is an analytic function such that $\operatorname{Lip} h \leqslant \frac{1}{2}(1+\operatorname{Lip}\varphi_1) < 1$, therefore, there exists an analytic function ψ such that $u = \psi(x,\tau)$, and then the stable invariant manifold of the periodic orbit labeled by $(\alpha_\infty, \rho_\infty)$ can be represented as the graph of

$(y, \theta, \rho, \alpha) = \tilde{\varphi}(x, \tau)$ with

$$\begin{aligned}
\tilde{\varphi}_1(x,\tau) &= \frac{\psi(x,\tau) - \varphi_1(\psi(x,\tau),\tau)}{2},\\
\tilde{\varphi}_2(x,\tau) &= \varphi_2(\psi(x,\tau),\tau),\\
\tilde{\varphi}_3(x,\tau) &= \frac{1}{1-\varphi_2^2(\psi(x,\tau),\tau)}(\rho_\infty + x\varphi_3(\psi(x,\tau),\tau)),\\
\tilde{\varphi}_4(x,\tau) &= \alpha_\infty + x\varphi_4(\psi(x,\tau),\tau) - 2y\tilde{\varphi}_3(x,\tau),
\end{aligned}$$

where (x, τ) belongs to a complex neighborhood of $(0, x_0) \times (0, \tau_0)$.

In order to prove the existence of an unstable invariant manifold of the periodic orbit labeled by α_∞ and ρ_∞ for the system (5.2) we perform the change given by $s = -t$ and

$$(\bar{x}, \bar{y}, \bar{\tau}, \bar{\theta}, \bar{\alpha}, \bar{\rho}, \bar{f}) = (x, -y, -\tau, \theta, -\alpha, \rho, -f).$$

In these new variables the dominant terms of system (5.2) do not change and therefore we can make the same argument as for the stable manifold and to conclude that there exists a two-dimensional unstable manifold associated to the periodic orbit of the system (5.2) labeled by α_∞ and ρ_∞.

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References

[1] M. Álvarez, J. Llibre, Heteroclinic orbits and Bernoulli shift for the elliptic collision restricted three-body problem, Arch. Rational Mech. Anal. 156 (2001) 317–357.
[2] J. Casasayas, E. Fontich, A. Nunes, Invariant manifolds for a class of parabolic points, Nonlinearity 5 (1992) 1193–1210.
[3] J. Delgado, C. Vidal, The tetrahedral 4-body problem, J. Dynamics Differential Equations 11 (4) (1999) 735–780.
[4] R. Easton, Parabolic orbits for the planar tree body problem, J. Differential Equations 52 (1984) 116–124.
[5] R. Easton, R. McGehee, Homoclinic phenomena for orbits doubly asymptotic to an invariant three-sphere, Indiana Univ. Math. J. 28 (1979) 211–240.
[6] N. Fenichel, Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J. 21 (1971/1972) 193–226.
[7] E. Fontich, Stable curves asymptotic to a degenerate fixed point, Nonlinear Anal. 35 (1999) 711–733.

[8] M. Hirsch, C.C. Pugh, Stable Manifolds and Hyperbolic Sets, in: Proceedings of the Symposium in Pure Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 1970, pp. 133–164.
[9] M. Hirsch, C.C. Pugh, M. Shub, Invariant Manifolds, in: Lecture Notes in Mathematics, Vol. 535, Springer, Berlin, 1977.
[10] M.C. Irwin, On the stable manifold theorem, Bull. London Math. Soc. 2 (1970) 196–198.
[11] N.G. Lloyd, Degree theory, in: Cambridge Tracts in Mathematics, Vol. 73, Cambridge University Press, Cambridge, New York, Melbourne, 1978.
[12] R. Martínez, C. Pinyol, Parabolic orbits in the elliptic restricted three body problem, J. Differential Equations 111 (1994) 299–339.
[13] R. McGehee, A stable manifold theorem for degenerate fixed points with applications to Celestial mechanics, J. Differential Equations 14 (1973) 70–88.
[14] R. Moeckel, Heteroclinic phenomena in the isosceles three-body problem, SIAM J. Math. Anal. 15 (1984) 857–876.
[15] C. Robinson, Homoclinic orbits and oscillations for the planar three-body problem, J. Differential Equations 52 (1984) 356–377.
[16] D.L. Slotnick, Asymptotic behavior of solutions of canonical systems near a closed, unstable orbit, in: S. Lefscetz (Ed.), Contributions to the Theory of Nonlinear Oscillations, Vol. IV, Annals of Mathematical Studies, Vol. 41, Princeton University Press, 1958, pp. 85–110.
[17] Z. Xia, Melnikov method and transversal homoclinic orbits in the restricted three-body problem, J. Differential Equations 96 (1992) 170–184.