THE RETURN TIMES PROPERTY ON LOGARITHM-TYPE SPACES

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ABSTRACT. Given a dynamical system $(\Omega, \Sigma, \mu, \tau)$ with μ a non-atomic probability measure and τ an invertible measure preserving ergodic transformation, we prove that the maximal operator, considered by I. Assani, Z. Buczolich and R. D. Mauldin in 2005,

$$N^* f(x) = \sup_{\alpha > 0} \alpha \# \left\{ k \geq 1 : \frac{|f(\tau^k x)|}{k} > \alpha \right\}$$

satisfies that

$$N^* : [L \log_3 L(\mu)] \longrightarrow L^{1,\infty}(\mu)$$

is bounded where the space $[L \log_3 L(\mu)]$ is defined by the condition

$$\|f\|_{[L \log_3 L(\mu)]} = \int_0^1 \frac{\sup_{t \leq y} t f_\mu^*(t)}{y} \log_3 \frac{1}{y} dy < \infty,$$

with $\log_3 x = 1 + \log_+ \log_+ \log_+ x$ and f_μ^* the decreasing rearrangement of f with respect to μ. This space is near $L \log_3 L(\mu)$, which is the optimal Orlicz space on which such boundedness can hold. As a consequence, the space $[L \log_3 L(\mu)]$ satisfies the Return Times Property for the Tail; that is, for every $f \in [L \log_3 L(\mu)]$, there exists a set X_0 so that $\mu(X_0) = 1$ and, for all $x_0 \in X_0$, all dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence

$$R_n g(y) = \frac{1}{n} f(\tau^n x_0) g(S^n y) \overset{n \to \infty}{\longrightarrow} 0, \qquad \nu\text{-a.e. } y \in Y.$$

1. Introduction and motivation

Let $(\Omega, \Sigma, \mu, \tau)$ be a dynamical system; that is, a probability measure space (Ω, Σ, μ) together with an invertible measure preserving transformation τ. In this context, the following return times theorem was proved in [10] (see also [8], [9] and [25]):

Theorem 1.1. *For every $1 \leq p \leq \infty$ and every $f \in L^p(\mu)$, there is a set $\Omega_f \subset \Omega$ with $\mu(\Omega_f) = 1$, such that, for any other dynamical system $(\Omega', \Sigma', \mu', \varrho)$, $g \in L^q(\mu')$ (with $\frac{1}{p} + \frac{1}{q} = 1$), and $x \in \Omega_f$, the sequence of means*

$$\frac{1}{n} \sum_{k=1}^{n} f(\tau^k x) g(\varrho^k y)$$

converges μ'-almost everywhere.

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The question then was to understand whether the fact that f and g lie in dual spaces was a necessary assumption for this theorem to hold (see [5, 17]). In an attempt to break this duality, in [3] and [4] it was proved that, given a dynamical system $(\Omega, \Sigma, \mu, \tau)$, if $f \in L^p(\mu)$ for some $p > 1$ (or even if $f \in L \log L(\mu)$), there is a set $\Omega_f \subset \Omega$ with $\mu(\Omega_f) = 1$ satisfying that for every sequence $(X_k)_k$ of i.i.d. random variables on a probability space (Ω', Σ', ν) with $X_k \in L^1(\nu)$ and any $x \in \Omega_f$,

$$\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x) X_k$$

converges ν-almost everywhere. However, for a general function in $L^1(\mu)$ this is no longer true [6]. One way to prove this negative result was studying the so called Return Times Property for the Tail (RTP) since it is easy to see that if the Return Times Theorem (RTT) holds for a function f, then the RTP also holds for this function f. This property is the following:

Definition 1.2. *A function f satisfies the RTP ($f \in RTP$) if, there exists a set X_0 so that $\mu(X_0) = 1$ and, for all $x_0 \in X_0$, all dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence*

$$R_n g(y) = \frac{1}{n} f(\tau^n x_0) g(S^n y)$$

converges to zero for ν-almost every $y \in Y$.

If X is a space such that $f \in RTP$ for every $f \in X$, we shall say that X satisfies the RTP or simply write $X \in RTP$. In particular,

$$X \in RTT \qquad \Longrightarrow \qquad X \in RTP.$$

In order to study the Return Times Theorem for $L^1(\mu)$ the following result was fundamental.

Theorem 1.3 ([4])**.** *Let $(c_n)_n$ be a sequence of nonnegative numbers such that $\lim_{n\to\infty} c_n = 0$. Then, the following two statements are equivalent.*

(a) $\sup_n \frac{1}{n}\#\{k : c_k > \frac{1}{n}\} < +\infty$.
(b) *For all finite dynamical systems $(Y, \mathcal{C}, \nu, S)$ and all $g \in L^1(\nu)$, the sequence $c_n g(S^n y)$ converges to zero for ν-almost every $y \in Y$.*

From this result, the following operator will play a fundamental role in this theory.

Definition 1.4. *We define the operator*

$$N^* f(x) = \sup_{\alpha > 0} \alpha N_\alpha f(x).$$

where, for $\alpha > 0$,

$$N_\alpha f(x) = \#\left\{k \geq 1 : \frac{|f(\tau^k x)|}{k} > \alpha\right\}.$$

Now, given $f \in L^1(\mu)$, it is known that the sequence $c_n = \frac{f(\tau^n x)}{n}$ converges to zero a.e. x, and hence,

$$f \in RTP \qquad \Longleftrightarrow \qquad \sup_{n\in\mathbb{N}} \frac{1}{n} N_{\frac{1}{n}} f(x) < +\infty \quad \text{a.e. } x.$$

It was proved in [6] that if the measure space is nonatomic and τ is ergodic, there exists $f \in L^1(\mu)$ such that $N^*f(x) = +\infty$ almost everywhere, and consequently, the Return Times Property for the Tail and the Return Times Theorem do not hold for $L^1(\mu)$ functions; that is

$$L^1(\mu) \notin RTP, \qquad \text{and} \qquad L^1(\mu) \notin RTT.$$

Taking all this into account, we can conclude that the study of the finiteness of N^*f is a key point in the Return Times theorems. On the other hand, it was proved in [18] that

$$N^* : L \log_2 L \longrightarrow L^{1,\infty}$$

is bounded and hence $L \log_2 L(\mu) \in RTP$. In the same paper, it was also stated that N^* cannot be bounded in any Orlicz space strictly larger that $L \log_3 L$, leaving as an open question the boundedness of

$$N^* : L \log_3 L(\mu) \longrightarrow L^{1,\infty}(\mu).$$

In this paper, we shall obtain the RTP for a logarithmic space near the endpoint $L \log_3 L(\mu)$, defined as follows:

Definition 1.5. *([15]) The space $[L \log_3 L(\mu)]$ is the set of measurable functions so that*

$$\|f\|_{[L \log_3 L(\mu)]} = \int_0^1 \frac{\sup_{t \le y} t f_\mu^*(t)}{y} \log_3 \frac{1}{y} dy < \infty,$$

where f_μ^* is the decreasing rearrangement of f with respect to μ defined by

$$f_\mu^*(t) = \inf\{y > 0 : \lambda_f^\mu(y) \le t\}, \qquad \lambda_f^\mu(t) := \nu(\{x \in \Omega : |f(x)| > t\}).$$

Here, and in the rest of this paper, we use the following notation:

$$\log_1(x) = 1 + \log_+(x) \quad \text{and} \quad \log_k(x) = \log_1 \log_{k-1}(x), \text{ for } k > 1.$$

Clearly,

$$[L \log_3 L] \subset L \log_3 L,$$

and the embedding is strict ([15]) since the fundamental functions of both spaces are not equivalent. However, if $h(y) = y f_\mu^*(y)$ is quasi-increasing in the sense that, there exists $C > 0$ so that

$$h(x) \le C h(y), \qquad \forall\ 0 < x \le y,$$

then

$$f \in [L \log_3 L(\mu)] \quad \Longleftrightarrow \quad f \in L \log_3 L(\mu).$$

On the other hand, in the context of Lorentz spaces (see definition below) it was proved in [14] that, for every $0 < q < 1$,

$$N^* : L^{1,q}(\mu) \longrightarrow L^{1,\infty}(\mu)$$

is bounded and hence

$$L^{1,q} \in RTP, \qquad \forall q < 1.$$

Observe that if we consider a measurable function g so that

$$g_\mu^*(t) = \frac{1}{t \log_1 \frac{1}{t} \log_2 \frac{1}{t} \big(\log_3 \frac{1}{t} \big)^3},$$

we have that $tg_\mu^*(t)$ is increasing and $g \in [L\log_3 L]$. However, g is not in either $L\log_2 L(\mu)$ or $L^{1,q}$, for any $0<q<1$. Therefore, the RTP of the space $[L\log_3 L]$ cannot not be deduced from the already known property for the spaces $L\log_2 L(\mu)$ or $L^{1,q}$. Moreover, once we prove our result, and as far as we know, the largest set of functions for which the RTP is satisfied will be up to now,

$$L\log_2 L(\mu) + [L\log_3 L(\mu)] + \bigcup_{0<q<1} L^{1,q}(\mu).$$

Before going on, let us recall the definition of the spaces which are going to be important to us. Given $0<p,q\le\infty$ and ν an arbitrary measure, $L^{p,q}(\nu)$ is the Lorentz space defined as the set of measurable functions such that

$$||f||_{L^{p,q}(\nu)} = \left(q\int_0^\infty y^{q-1}\lambda_f^\nu(y)^{\frac{q}{p}}dy\right)^{1/q} = \left(\frac{q}{p}\int_0^\infty f_\nu^*(t)^q t^{\frac{q}{p}-1}dt\right)^{1/q} < \infty, \quad \text{if } q<\infty,$$

and if $q=\infty$,

$$||f||_{L^{p,\infty}(\nu)} = \sup_{y>0} y\lambda_f^\nu(y)^{1/p} = \sup_{t>0} t^{1/p} f_\nu^*(t) < \infty.$$

If ν is the counting measure on $\mathbb{Z}$, we shall write $L^{p,q}(\nu) = \ell^{p,q}$. Recall also that $L\log L(\nu) \subsetneq L^1(\nu)$ is the space of ν-measurable functions such that

$$\|f\|_{L\log L(\nu)} = \int_0^\infty f_\nu^*(t)\log_1\frac{1}{t}dt < \infty.$$

We will also need to consider other log-type spaces, such as $L\log L\log_3 L$ and $L\log_3 L$ defined, respectively, as the set of μ-measurable functions such that

$$\|f\|_{L\log L\log_3 L(\nu)} = \int_0^\infty f_\nu^*(t)\log_1\frac{1}{t}\log_3\frac{1}{t}dt < \infty,$$

and

$$\|f\|_{L\log_3 L(\nu)} = \int_0^\infty f_\nu^*(t)\log_3\frac{1}{t}dt < \infty.$$

Using standard techniques of discretization and transference theory, it turns out that the boundedness of the operator N^* is formally equivalent to the boundedness of the following apparently easier operator defined on the interval $(0,1)$. This operator will be fundamental for our purposes.

Definition 1.6. *([3, 22]) We define the operator*

$$Af(x) = \left\|\frac{f\chi_{(0,x)}}{x-\cdot}\right\|_{L^{1,\infty}(0,1)} = \sup_{\lambda>0}\lambda\left|\left\{0<y<x : |f(y)| > \lambda(x-y)\right\}\right|.$$

Hence, the study of boundedness properties of the operator A became an interesting question. In fact, we are going to start by proving boundedness properties of A, since the ideas to solve the problem for N^* come from this operator.

To prove our main results we are going to use the theory of Muckenhoupt weights and some of the ingredients behind the extrapolation theorem of Rubio de Francia [24], together with some results connected with the theory of Yano's extrapolation and its several recent extensions. Also, we should recall that by Stein and Weiss lemma [26] we have that, there exists $C>0$ so that, for every $(c_n)_n$ of positive numbers so that $\sum_n c_n = 1$, and any collection of functions $(g_n)_n$,

$$\Big\| \sum_{n=0}^{\infty} c_n g_n \Big\|_{L^{1,\infty}} \le C \sum_{n=0}^{\infty} c_n \log \frac{1}{c_n} \|g_n\|_{L^{1,\infty}}. \tag{1.1}$$

Form here, we have, taking $c_n = \frac{1}{(n+1)^2}$ and $f_n = (n+1)^2 g_n$, that

$$\Big\| \sum_{n=0}^{\infty} f_n \Big\|_{L^{1,\infty}} \le C \sum_{n=0}^{\infty} \log(n+2) \|f_n\|_{L^{1,\infty}}. \tag{1.2}$$

Finally, in what follows, we shall assume that τ is an ergodic transformation.

The paper is organized as follows: in Section 2 we present all the previous results and technical questions that we need and, in Section 3, our main results will be developed.

From now on, we write $x \lesssim y$ when there is a positive constant $C>0$ such that $x \le Cy$. If both $x \lesssim y$ and $y \lesssim x$, then we write $x \approx y$. The constants involved do not depend on any parameter that is not fixed in its context.

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2. Previous Lemmas on interpolation, extrapolation and weighted theory

2.1. **Weighted theory.** The starting result is related to the Hardy-Littlewood maximal operator M defined on locally integrable functions on $\mathbb{R}$ by

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| dy,$$

where the supremum is taken over all intervals $I \subseteq \mathbb{R}$ containing x. We recall that a positive measurable function w is called a weight if $w \in L^1_{loc}(\mathbb{R})$. In this case, $L^{p,q}(w) = L^{p,q}(\nu)$ with $d\nu = w dx$ and, for any measurable set F, we write $w(F) = \int_F w(x) dx$. If $w=1$, we simply write $|F|$.

Proposition 2.1 ([16, 20])**.**

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded if and only if $w \in A_p^{\mathcal{R}}$*; that is,*

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{E \subset I} \frac{|E|}{|I|} \Big(\frac{w(I)}{w(E)} \Big)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taken over all intervals I and every measurable set $E \subset I$. Moreover,

$$\|M\|_{L^{p,1}(w) \to L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}}. \tag{2.2}$$

Clearly

$$A_p^{\mathcal{R}} \subset A_q^{\mathcal{R}} \quad \text{for every } p \le q, \text{ and } \|w\|_{A_q^{\mathcal{R}}} \le \|w\|_{A_p^{\mathcal{R}}}.$$

These classes of weights are closely related to the A_p class introduced by Muckenhoupt in [21]. For our purposes, a fundamental property of $A_p^{\mathcal{R}}$ is the following [13].

Lemma 2.2. *There exists a constant $C>0$ so that, for all $p\geq 1$,*
$$\|(Mf)^{1-p}\|_{A_p^{\mathcal{R}}}\leq C, \qquad \forall f\in L^1_{loc}(\mathbb{R}).$$

We shall need a discretized version of Lemma 2.2 and to this end we have to define first the class of weights $A_p^{\mathcal{R}}(\mathbb{Z})$. In fact, as shown in [16], condition (2.1) is equivalent to
$$\|\chi_I\|_{L^{p,1}(w)}\|w^{-1}\chi_I\|_{L^{p',\infty}(w)}\lesssim |I|$$
for every interval I, and we use this characterization to define $A_p^{\mathcal{R}}(\mathbb{Z})$ as follows:

Definition 2.3. *A sequence $u=(u_i)_{i\in\mathbb{Z}}\in A_p^{\mathcal{R}}(\mathbb{Z})$ if*
$$\left(\sum_{i=j}^{k}u_i\right)\sup_{t>0}t^p\left[\sum_{i=j,u_i<\frac{1}{t}}^{k}u_i\right]^{p-1}\lesssim (k-j)^p, \qquad \forall j<k\in\mathbb{Z}.$$

Set the discrete maximal operator defined for sequences $a=(a_i)_{i\in\mathbb{Z}}$ by
$$M_d a(k)=\sup_n\frac{1}{2n+1}\sum_{i=-n}^{n}|a_{i+k}|, \qquad k\in\mathbb{Z}.$$

Lemma 2.4. *There exists a constant $C>0$ so that, for every sequence $a\neq 0$,*
$$\|(M_d a)^{1-p}\|_{A_p^{\mathcal{R}}(\mathbb{Z})}\leq C.$$

Proof. By definition of the class $A_p^{\mathcal{R}}(\mathbb{Z})$, we have to prove that, for every $i<j$,
$$B_j:=\left(\sum_{k=i}^{j}(M_d a)^{1-p}(k)\right)\sup_{t>0}t^p\left[\sum_{k=i,(M_d a)^{1-p}(k)<\frac{1}{t}}^{j}(M_d a)^{1-p}(k)\right]^{p-1}\lesssim (j-i)^p.$$

Now,
$$\begin{aligned}B_j &= \left(\sum_{k=i}^{j}(M_d a)^{1-p}(k)\right)\sup_{t>0}\left[t^p\sum_{k=i,(M_d a)(k)>t}^{j}(M_d a)^{1-p}(k)\right]^{p-1}\\ &\lesssim \left(\sum_{k=i}^{j}(M_d a)^{1-p}(k)\right)\sup_{t>0}\left[t\sharp\{i\leq k\leq j:(M_d a)(k)>t\}\right]^{p-1}.\end{aligned}$$

We can assume without loss of generality that $j\geq 0$. Let us write $a=\underline{a}^{i,j}+\bar{a}^{i,j}$, where
$$\underline{a}^{i,j}=(\cdots,0,0,a_{2i-j},a_{2i-j+1},\cdots,a_j,\cdots,a_{2j-i},0,0,\cdots).$$

Then, if $i\leq k,k'\leq j$, it is clear that $(M_d\bar{a}^{i,j})(k)\approx(M_d\bar{a}^{i,j})(k')$ and hence there exists a universal constant $B>0$ so that
$$(M_d\bar{a}^{i,j})(k)\leq B\min_{i\leq s\leq j}(M_d a)(s), \qquad \forall i\leq k\leq j.$$

Set $\beta_{i,j}:=\min_{i\leq s\leq j}(M_d a)(s)$. Then, if $t\leq 2B\beta_{i,j}$,
$$t\sharp\{i\leq k\leq j:(M_d a)(k)>t\}\lesssim (j-i)\beta_{i,j},$$

and hence

$$\left(\sum_{k=i}^{j}(M_d a)^{1-p}(k)\right)\left[t\sharp\{i\le k\le j:(M_d a)(k)>t\}\right]^{p-1}\lesssim (j-i)^p. \tag{2.3}$$

On the other hand, if $t>2B\beta_{i,j}$, using that $M_d:\ell^1\to\ell^{1,\infty}$ we have,

$$t\sharp\{i\le k\le j:(M_d\underline{a}^{i,j})(k)>t/2\}\lesssim\sum_{k=2i-j}^{2j-i}|a_k|\lesssim (j-i)\beta_{i,j},$$

and

$$t\sharp\{i\le k\le j:(M_d\bar{a}^{i,j})(k)>t/2\}\le t\sharp\{i\le k\le j:B\beta_{i,j}>t/2\}=0,$$

therefore,

$$\begin{aligned} t\sharp\{i\le k\le j:(M_d a)(k)>t\} &\le t\sharp\{i\le k\le j:(M_d\underline{a}^{i,j})(k)>t/2\}\\ &+ t\sharp\{i\le k\le j:(M_d\bar{a}^{i,j})(k)>t/2\}\le (j-i)\beta_{i,j},\end{aligned}$$

and the result follows as in (2.3). □

In the context of our dynamical system, the following result was proved in [23].

Theorem 2.5. *The ergodic maximal operator*

$$M_\tau f(x)=\sup_{n\in\mathbb{N}}\frac{1}{2n+1}\sum_{i=-n}^{n}|f(\tau^i x)|$$

satisfies that

$$M_\tau : L^{p,1}(ud\mu)\longrightarrow L^{p,\infty}(ud\mu)$$

if, for a.e. x, the sequence $u^x=(u(\tau^i x))_i$ satisfies that $u^x\in A_p^{\mathcal{R}}(\mathbb{Z})$ uniformly in x; that is, there exists $C>0$, so that

$$\left(\sum_{i=0}^{k}u(\tau^i x)\right)\sup_{t>0}t^p\left[\sum_{i=0,u(\tau^i x)<\frac{1}{t}}^{k}u(\tau^i x)\right]^{p-1}\le Ck^p,\qquad a.e.\ x.$$

As a consequence of Lemma 2.4 and Theorem 2.5 we finally obtain the following result.

Corollary 2.6. *Given a dynamical system (Ω,Σ,μ,τ), $1<p<\infty$, h a measurable positive function and $u=(M_\tau h)^{1-p}$, we have that the ergodic maximal operator*

$$M_\tau : L^{p,1}(ud\mu)\longrightarrow L^{p,\infty}(ud\mu)$$

is bounded with constant independent of h.

2.2. **Interpolation.** The following lemma is classical (see for example [7]) and the computation of the constant, which is fundamental for our purposes, was done in detail in [12].

Lemma 2.7. *Let T be a quasi-sublinear operator such that, for some weight u,*

$$T : L^{q_j,1}(u) \longrightarrow L^{q_j,\infty}(u)$$

is bounded with constant M_j, for $j = 0, 1$. Then, for every $0 < \theta < 1$, if $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, we have that

$$T : L^{q,\infty}(u) \longrightarrow L^{q,\infty}(u)$$

is bounded with constant controlled by

$$\frac{C(q_1 - q_0)}{(q - q_0)(q_1 - q)} M_0^{1-\theta} M_1^{\theta}.$$

2.3. **Yano's extrapolation type results.** The problem of approaching the endpoint L^1 only from information in L^p for $p > 1$ is the starting point of another extrapolation theory, initiated by S. Yano in 1951, [27]:

Theorem 2.8. *Let ν be a finite measure and $p_0 > 1$. If T is a sublinear operator such that, for every $1 < p \leq p_0$,*

$$T : L^{p,1}(\nu) \longrightarrow L^p(\nu)$$

is bounded with norm controlled by $C(p-1)^{-1}$ for some $C > 0$ independent of p, then,

$$T : L \log L(\nu) \longrightarrow L^1(\nu).$$

There is also a Yano's extrapolation theorem concerning weak-type spaces. In 1996, N. Yu Antonov [1] proved that there is almost everywhere convergence for the Fourier series of every function in $L \log L \log_3 L(\mathbb{T})$. Even though he did not write it explicitly, behind his ideas there was the following extrapolation argument (see [2, 11] for more details):

Theorem 2.9. *Let ν be a finite measure and $p_0 > 1$. If T is a sublinear operator such that, for every $1 < p \leq p_0$,*

$$T : L^{p,1}(\nu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with norm controlled by $C(p-1)^{-1}$, then,

$$T : L \log L \log_3 L(\nu) \longrightarrow L^{1,\infty}(\nu).$$

Finally, a second variant of Yano's extrapolation theorem was done in [15]. We present here a new and much shorter proof of this result.

Theorem 2.10. *Let ν be a finite, non-atomic measure and $p_0 > 1$. If T is a sublinear operator such that, for every $1 < p \leq p_0$,*

$$T : L^{p,\infty}(\nu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with norm controlled by $C(p-1)^{-1}$, then,

$$T : [L \log_3 L(\nu)] \longrightarrow L^{1,\infty}(\nu),$$

where $[L \log_3 L(\nu)]$ is the set of measurable functions such that

$$\|f\|_{[L \log_3 L(\nu)]} = \int_0^1 \frac{\sup_{t \leq y} t f_\nu^*(t)}{y} \log_3 \frac{1}{y} dy < \infty.$$

Proof. We shall assume, for simplicity, that ν is a probability measure. Let f be a ν-measurable function so that $\|f\|_\infty \leq 1$, then for every $1 < p \leq p_0$ and every $0 < t \leq 1$,

$$t^{1/p}(Tf)^*_\nu(t) \lesssim \frac{1}{p-1}\sup_{s>0} s^{1/p} f^*_\nu(s) \leq \frac{1}{p-1}(\sup_{s>0} s f^*_\nu(s))^{1/p} = \frac{\|f\|^{1/p}_{L^{1,\infty}}}{p-1},$$

and hence, for every $0 < t \leq 1$,

$$(Tf)^*_\nu(t) \lesssim \inf_{1<p\leq p_0} \frac{1}{p-1}\bigg(\frac{\|f\|_{L^{1,\infty}}}{t}\bigg)^{1/p}.$$

Now, since $\inf_{1<p<p_0} \frac{A^{1/p}}{p-1} \lesssim A(1+\log_1 \frac{1}{A})$, we have that

$$(Tf)^*_\nu(t) \lesssim \varphi\bigg(\frac{\|f\|_{L^{1,\infty}}}{t}\bigg), \qquad \varphi(y) = y\log_1 \frac{1}{y}.$$

From here, it follows that

$$\|Tf\|_{L^{1,\infty}} \lesssim \varphi(\|f\|_{L^{1,\infty}}), \qquad ||f||_\infty \leq 1.$$

Now, given a measurable function f, we know [7] that there exists a measure preserving transformation σ from our measure space to $(0,1)$ so that $f(x) = f^*_\nu(\sigma(x))$. Hence, if we write $I_n = (2^{1-2^{n+1}}, 2^{1-2^n})$,

$$f^*_\nu(s) = \sum_{n=0}^\infty f^*_\nu(s)\chi_{I_n}(s),$$

and thus

$$f(x) = f^*_\nu(\sigma(x)) = \sum_{n=0}^\infty f^*_\nu(\sigma(x))\chi_{I_n}(\sigma(x)) = \sum_{n=0}^\infty f(x)\chi_{E_n}(x) := \sum_{n=0}^\infty f_n(x),$$

where $\|f_n\|_\infty \leq f^*_\nu(2^{1-2^{n+1}})$. Hence, if $\tilde{f}_n(x) = f_n(x)/f^*_\nu(2^{1-2^{n+1}})$, we have that $\|\tilde{f}_n\|_\infty \leq 1$ and

$$Tf(x) \leq \sum_{n=0}^\infty (Tf_n)(x) = \sum_{n=0}^\infty f^*_\nu(2^{1-2^{n+1}})(T\tilde{f}_n)(x).$$

From here, using (1.2), we obtain that

$$\begin{aligned}
\|Tf\|_{L^{1,\infty}} &\lesssim \sum_{n=0}^{\infty} \log(n+2) f_\nu^*(2^{1-2^{n+1}}) \|T\tilde{f}_n)\|_{L^{1,\infty}} \\
&\lesssim \sum_{n=0}^{\infty} \log(n+2) f_\nu^*(2^{1-2^{n+1}}) \varphi(\|\tilde{f}_n\|_{L^{1,\infty}}) \\
&\lesssim \sum_{n=0}^{\infty} \log(n+2) f_\nu^*(2^{1-2^{n+1}}) \varphi\left(\frac{\|f_n\|_{L^{1,\infty}}}{f_\nu^*(2^{1-2^{n+1}})}\right) \\
&\lesssim \sum_{n=0}^{\infty} \log(n+2) f_\nu^*(2^{1-2^{n+1}}) \varphi\left(\frac{\sup_{t\in I_n} t f_\nu^*(t)}{f_\nu^*(2^{1-2^{n+1}})}\right) \\
&\approx \sum_{n=0}^{\infty} \log(n+2) \sup_{t\in I_n} t f_\nu^*(t) \log_1 \frac{f_\nu^*(2^{1-2^{n+1}})}{\sup_{t\in I_n} t f_\nu^*(t)} \\
&\lesssim \sum_{n=0}^{\infty} \log(n+2) 2^n \sup_{t\in I_n} t f_\nu^*(t) \lesssim \|f\|_{[L\log_3 L(\nu)]},
\end{aligned}$$

and the result follows. □

Remark 2.11. *Observe that the theorem above is still true if we change the sublinearity condition on T by the condition that T is sublinear on functions with pairwise disjoint supports.*

3. Main results

As mentioned in the introduction, we are going to start by proving boundedness properties of the operator A, since the ideas to solve the case of N^* are exactly of the same nature. We shall extend the operator A to the whole real line; that is, we shall consider

$$\bar{A}f(x) = \left\| \frac{f\chi_{(-\infty,x)}}{x-\cdot} \right\|_{L^{1,\infty}(\mathbb{R})}, \qquad \forall f \in L^1_{loc}(\mathbb{R}).$$

It was proved in [18] that

$$A : L\log_2 L(0,1) \longrightarrow L^{1,\infty}(0,1)$$

is bounded and it was also stated that A cannot be bounded on any Orlicz space strictly bigger that $L\log_3 L(0,1)$, leaving as an open question the boundedness of

$$A : L\log_3 L(0,1) \longrightarrow L^{1,\infty}(0,1). \tag{3.1}$$

That is, in the scale of logarithmic spaces X, $L\log_3 L(0,1)$ would be the optimal one satisfying that $A : X \to L^{1,\infty}(0,1)$ is bounded. In our first attempt to solve the conjectured boundedness in (3.1), the following observation was made in [15]: given a locally integrable function f, it is known [7] that

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)ds.$$

Therefore,

$$||Mf||_{L\log_3 L} \approx ||f||_{L\log L\log_3 L}.$$

and hence, if we prove that

$$A \circ M : L^p \longrightarrow L^{p,\infty} \tag{3.2}$$

is bounded with constant $C/(p-1)$, by Theorem 2.9 we have that

$$A \circ M : L \log L \log_3 L \longrightarrow L^{1,\infty}$$

will be bounded. Thus,

$$||Ag||_{L^{1,\infty}} \lesssim ||g||_{L \log_3 L}$$

when g belongs to the collection of functions of the form $g = Mf$. Since $M : L^p \longrightarrow L^{p,\infty}$ is bounded with a uniform constant independent of p, (3.2) will hold if

$$A : L^{p,\infty} \longrightarrow L^{p,\infty}, \qquad \frac{C}{p-1};$$

that is

$$\|Af\|_{L^{p,\infty}} \lesssim \frac{1}{p-1}\|f\|_{L^{p,\infty}}.$$

This question remained open in [15] and its proof is one of our main results. We first emphasize the following properties of the operator A:

(1) A is not a sublinear operator, but it is quasi-linear; that is, for some $C > 1$,

$$A(f+g) \leq C(Af + Ag),$$

and hence the interpolation Lemma 2.7 can be applied.

(2) If $(f_n)_n$ is a collection of functions with pairwise disjoint supports, we do have sublinearity; that is

$$A(\sum_n f_n) \leq \sum_n A(f_n),$$

and hence, by Remark 2.11, Theorem 2.10 can be applied.

Lemma 3.1. *For every $0 \leq \delta < 1$ and every $h \in L^1_{loc}(\mathbb{R})$, $u = (Mh)^\delta$,*

$$\bar{A} : L^{2,\infty}(u^{-1}) \longrightarrow L^{2,\infty}(u^{-1})$$

is bounded with constant less than or equal to $\frac{C}{1-\delta}$ with C independent of h.

Proof. A simple calculation shows that, for every measurable set E,

$$\bar{A}\chi_E \leq M\chi_E,$$

and hence, by (2.2), for every $q \geq 1$ and every $v \in A^{\mathcal{R}}_q$,

$$\|\bar{A}\chi_E\|_{L^{q,\infty}(v)} \leq \|v\|_{A^{\mathcal{R}}_q} v(E)^{1/q}.$$

By Lemma 2.2, we have that, for every $r \geq 1+\delta$, $u^{-1} \in A^{\mathcal{R}}_r$ with $\|v\|_{A^{\mathcal{R}}_r} \lesssim 1$, and hence

$$\bar{A} : L^{r,1}(u^{-1}) \longrightarrow L^{r,\infty}(u^{-1}), \qquad \frac{C}{1-r}. \tag{3.3}$$

Finally, we use Lemma 2.7 with (3.3), $r = \max(1.5, 1+\delta)$ and $r = 3-\delta$, to conclude that

$$\bar{A} : L^{2,\infty}(u^{-1}) \longrightarrow L^{2,\infty}(u^{-1})$$

with constant controlled by $\frac{C}{1-\delta}$, as we wanted to see. $\square$

Theorem 3.2. *For every* $1 < p \leq 2$,
$$\bar{A} : L^{p,\infty}(\mathbb{R}) \longrightarrow L^{p,\infty}(\mathbb{R})$$
is bounded with constant less than or equal to $C(p-1)^{-1}$.

Proof. By density, we can assume without lost of generality that $f \in L^{p,\infty}(\mathbb{R})$ satisfies $\lim_{t\to 0} \sup_{s\leq t} sf^*(s)^p = 0$. Then, if $F(t) = \sup_{s\leq t} sf^*(s)^p$ we have that F is an increasing function so that $F(t)/t$ is decreasing and hence F is quasi-concave and $F(0+) = 0$. Thus, there exists a decreasing function h so that
$$f^*(t)^p \leq \frac{\sup_{s\leq t} sf^*(s)^p}{t} \approx \frac{1}{t}\int_0^t h(x)dx \approx (Mh)^*(t).$$
From here it follows that
$$\|(Mh)^{1/p}\|_{L^{p,\infty}(\mathbb{R})} = \sup_{t>0} t^{1/p}(Mh)^*(t)^{1/p} = \Big(\sup_{t>0} t(Mh)^*(t)\Big)^{1/p} \approx \|f\|_{L^{p,\infty}(\mathbb{R})}.$$
Therefore, by Lemma 3.1,
$$\begin{aligned}
\lambda_{\bar{A}f}(y) &\leq \lambda_{(Mh)^{1/p}}(\gamma y) + \int_{\{(\bar{A}f)>y,(Mh)^{1/p}\leq\gamma y\}} 1dx \\
&\lesssim \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}} + \int_{\{(\bar{A}f)>y,(Mh)^{1/p}\leq\gamma y\}} \left(\frac{\gamma y}{(Mh)^{1/p}}\right)^{2-p} dx \\
&\leq \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}} + \frac{\gamma^{2-p}}{y^p}\sup_{z>0} z^2 \int_{\{(\bar{A}f)>z\}} (Mh(x))^{\frac{p-2}{p}} dx \\
&\lesssim \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}} + \frac{\gamma^{2-p}}{(p-1)^2 y^p}\sup_{z>0} z^2 \int_{\{|f|>z\}} (Mh(x))^{\frac{p-2}{p}} dx.
\end{aligned}$$
On the other hand, if F is a measurable set with $|F| < \infty$ and $\alpha \leq 0$, it is easy to see that, for almost every $0 < t < |F|$,
$$\big(g^\alpha \chi_F\big)^*(t) = \big((g\chi_F)^*(|F| - t)\big)^\alpha,$$
and hence, by Hardy's inequality,
$$\int_{\{|f|>z\}} (Mh(x))^{\frac{p-2}{p}} dx \leq \int_0^{\lambda_f(z)} \big((Mh)^*(\lambda_f(z) - t)\big)^{\frac{p-2}{p}} dt = \int_0^{\lambda_f(z)} (Mh)^*(t)^{\frac{p-2}{p}} dt.$$
As a consequence, since $(Mh)^*(t) \geq f^*(t)^p$,
$$\begin{aligned}
\lambda_{\bar{A}f}(y) &\leq \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}} + \frac{\gamma^{2-p}}{(p-1)^2 y^p}\sup_{z>0} z^2 \int_{\{f^*>z\}} (f^*(t))^{p-2} dt \\
&\leq \left(\frac{1}{\gamma^p} + \frac{\gamma^{2-p}}{(p-1)^2}\right)\|f\|^p_{L^{p,\infty}},
\end{aligned}$$
and the result follows taking $\gamma = (p-1)$ $\square$

Obviously, the same boundedness holds for the original operator A, and by Theorem 2.10 the following results can be deduced:

Corollary 3.3. *The operator A satisfies that, for every $1 < p \leq 2$,*

$$A : L^{p,\infty}(0,1) \longrightarrow L^{p,\infty}(0,1), \qquad \frac{C}{p-1},$$

and consequently,

$$A : [L \log_3 L(0,1)] \longrightarrow L^{1,\infty}(0,1).$$

Corollary 3.4. *If f is a positive function defined on $\mathbb{Z}$, and*

$$N^{*,d} f(i) = \sup_{\alpha>0} \alpha \, \#\left\{ k \geq 1 : \frac{f(i+k)}{k} > \alpha \right\},$$

we have that

$$N^{*,d} : \ell^{p,\infty} \longrightarrow \ell^{p,\infty}, \qquad \frac{C}{p-1}.$$

Proof. To see this, let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = f([x])$. It is then easy to check ([14]) that, if $x \in (i, i+1)$,

$$\#\left\{ k \geq 1 : \frac{f(i+k)}{k} > \alpha \right\} \leq \left| \left\{ y > 0 : \frac{F(x+y)}{y} > \frac{\alpha}{2} \right\} \right|,$$

and thus, $N^{*,d} f(i) \lesssim \bar{A} F(x)$, for every $x \in (i, i+1)$ from which the result follows using Theorem 3.2. □

In order to prove the RTP for the space $[L \log_3 L]$, our first idea was to use the operator $N^{*,d}$ together with standard techniques in transference theory, but we failed in proving the corresponding transference theorem due to the special structure of $L^{p,\infty}(\mu)$. However, using Corollary 2.6, we can reproduce the technique in the proof of our Theorem 3.2 to show our last main result:

Theorem 3.5. *Given a dynamical system $(\Omega, \Sigma, \mu, \tau)$, we have that:*

a) *For every $1 < p \leq 2$,*

$$N^* : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu), \qquad \frac{C}{p-1}.$$

b)

$$N^* : [L \log_3 L(\mu)] \longrightarrow L^{1,\infty}(\mu).$$

c)

$$[L \log_3 L(\mu)] \in RTP.$$

Proof. Clearly, c) is an immediate consequence of b) and b) of a) by Theorem 2.10. Hence, it only remains to prove a). The proof of a) follows the same pattern as the proof of Theorem 3.2 using the following facts:

(1) A simple calculation shows that, for every measurable set E,

$$N^* \chi_E \leq M_\tau \chi_E.$$

(2) Hence, by Corollary 2.6 we have that if h is a positive measurable function and $v = (M_\tau h)^{1-q}$ with $1 \leq q < 2$,

$$\| N^* \chi_E \|_{L^{r,\infty}(v)} \lesssim v(E)^{1/r}, \qquad \forall r \geq q.$$

(3) Since N^* is monotone and sublinear on functions with disjoint supports we have that the previous estimate can be extended to

$$N^* : L^{r,1}(v) \longrightarrow L^{r,\infty}(v), \qquad \forall r \geq q,$$

with constant independent of h.

(4) From here it follows, as in Lemma 3.1, that

$$N^* : L^{2,\infty}(v) \longrightarrow L^{2,\infty}(v), \qquad \frac{C}{2-q}.$$

(5) In [19] it was proved that

$$(M_\tau f)^*_\mu(t) \approx \frac{1}{t}\int_0^t f^*_\mu(s)ds.$$

Hence, the proof can be finished exactly as in Theorem 3.2 with the obvious modifications, which we include for the sake of completeness:

Take h so that

$$f^*_\mu(t)^p \leq \frac{\sup_{s\leq t} s f^*_\mu(s)^p}{t} \approx \frac{1}{t}\int_0^t h^*_\mu(s)ds \approx (M_\tau h)^*_\mu(t), \qquad \forall\, 0<t<1.$$

Notice that we are using the fact that μ is σ-finite and nonatomic, which ensures that, for every positive decreasing and right continuous function g on $(0,\infty)$, there exists a measurable function h so that $h^*_\mu(t) = g(t)$, for every $0<t<1$. (see [7]).

Hence,

$$\begin{aligned}
\lambda^\mu_{N^*f}(y) &\leq \lambda^\mu_{(M_\tau h)^{1/p}}(\gamma y) + \int_{\{N^*f>y,(M_\tau h)^{1/p}\leq\gamma y\}} 1 d\mu(x)\\
&\lesssim \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}(\mu)} + \int_{\{N^*f>y,(M_\tau h)^{1/p}\leq\gamma y\}} \left(\frac{\gamma y}{(M_\tau h)^{1/p}}\right)^{2-p} d\mu(x)\\
&\leq \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}(\mu)} + \frac{\gamma^{2-p}}{y^p}\sup_{z>0} z^2 \int_{\{N^*f>z\}} (M_\tau h(x))^{\frac{p-2}{p}} d\mu(x)\\
&\lesssim \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}(\mu)} + \frac{\gamma^{2-p}}{(p-1)^2y^p}\sup_{z>0} z^2 \int_{\{|f|>z\}} (M_\tau h(x))^{\frac{p-2}{p}} d\mu(x)\\
&\lesssim \frac{1}{y^p\gamma^p}\|f\|^p_{L^{p,\infty}(\mu)} + \frac{\gamma^{2-p}}{(p-1)^2y^p}\sup_{z>0} z^2 \int_{\{f^*_\mu>z\}} ((M_\tau h)^*_\mu(t))^{\frac{p-2}{p}} dt\\
&\leq \left(\frac{1}{\gamma^p} + \frac{\gamma^{2-p}}{(p-1)^2}\right)\|f\|^p_{L^{p,\infty}(\mu)},
\end{aligned}$$

and the result follows taking $\gamma = p-1$. □

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